QIC710/CS768/CO681/PH767/AM871 Introduction to Quantum Information Processing (F19)

Assignment 4
Due date: November 7, 2019

1. **Unitary operations as rotations on the Bloch sphere [12 points, 6 each].** It is known that every one-qubit unitary operation acts on the Bloch sphere as a rotation about some axis. For each of the following unitary operations, describe the axis of rotation (which can be an eigenvector of the unitary) and the angle of rotation on the Bloch sphere:

   (a) $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

   (b) $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$

2. **Expressing channels as mixed-unitary channels [12 points, 4 each].** Recall that a channel is *mixed-unitary* if it can be expressed in terms of Kraus operators $A_0, \cdots, A_{m-1}$ that are of the form $A_k = \sqrt{p_k}\, U_k$, where $U_0, \ldots, U_{m-1}$ are unitary and $(p_0, \ldots, p_{m-1})$ is a probability vector.

   (a) On the face of it, the one-qubit *decoherence* channel with Kraus operators, that we defined in class with Kraus operators $A_0 = |0\rangle\langle 0|$ and $A_1 = |1\rangle\langle 1|$, does not appear to be a mixed-unitary channel (since A_0 and A_1 are clearly not of the above form). However, the Kraus representation of a channel need not be unique. Show that there is *another* pair of Kraus operators B_0 and B_1 that also generates the decoherence channel and for which B_0 and B_1 are of the above form of a mixed-unitary channel.

   (b) Consider the one-qubit channel that maps every 2×2 density matrix to the density matrix $\left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right)$. Show that this is a mixed-unitary channel by exhibiting four Kraus operators A_0, A_1, A_2, A_3 for it in the form of a mixed-unitary channel.

   (c) Show that the channel in part (b) cannot be expressed as a *mixed-unitary* channel with only *three* Kraus operators.

3. **A channel that is not mixed unitary [12 points, 4 each].** Considering question 2, it may be tempting to conjecture that all channels whose Kraus operators are square matrices are mixed-unitary channels. Here we give a counterexample to this conjecture. Let

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   (a) Show that A_0 and A_1 are valid Kraus operators (i.e., that $A_0^\dagger A_0 + A_1^\dagger A_1 = I$).

   (b) Show that, for any 2×2 density matrix ρ, this channel maps ρ to the density matrix $|0\rangle\langle 0|$.

   (c) Prove that this is *not* a mixed unitary channel. (Hint: consider the effect of a mixed unitary channel on the maximally mixed state $\left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right)$.)

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4. **Expressing an AND gate as a quantum channel [12 points].** Recall the binary AND operation, denoted as $\wedge$, defined as

a	b	$a \wedge b$
0	0	0
0	1	0
1	0	0
1	1	1

Here we consider a channel that, for all $a, b \in \{0, 1\}$, maps the two-qubit state $|a, b\rangle$ to the one-qubit state $|a \wedge b\rangle$. Of course, no unitary operation can perform this mapping, since the input and output dimension do not match; however, a quantum channel can compute this mapping.

(a) [8 points] Give a sequence of 2×4 matrices $A_0, \ldots, A_{m-1}$ such that $\sum_{j=0}^{m-1} A_j^\dagger A_j = I$ that compute the AND operation in the sense that, for all $a, b \in \{0, 1\}$, when $\rho = |a, b\rangle\langle a, b|$,

$$\sum_{j=0}^{m-1} A_j \rho A_j^\dagger = |a \wedge b\rangle\langle a \wedge b|.$$

(b) [4] Your operation from part (a) maps all computational basis states to pure states. Does it map all pure input states to pure output states? Either prove the answer is yes, or provide a counterexample.

5. **A non-local game with ternary inputs [12 points].** Consider the non-local game where Alice and Bob receive inputs $s, t \in \{0, 1, 2\}$ respectively (generated randomly according the uniform distribution) and they must produce outputs $a, b \in \{0, 1\}$ respectively, where the winning condition is that: if $s = t$ then $a = b$; and if $s \neq t$ then $a \neq b$.

(a) [5 points] Show that any classical strategy (i.e., without shared entanglement) can achieve success probability at most $7/9 = 0.777\ldots$ for this game. In your proof, you may assume that the classical strategy is deterministic (it can be shown that the same bound for probabilistic strategies follows from this).

(b) [5] Show that there is a strategy using a shared entangled state that has the following property: if $s = t$ then $a = b$; if $s \neq t$ then $\Pr[a \neq b] \geq 3/4$.

(c) [2] Deduce that the strategy in part (b) succeeds at the game with probability at least $5/6 = 0.833\ldots$.