XOR Games at Full Tilt and the Hardness of Binary Nonlocal Games

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Abstract

We show that a variant of the XOR nonlocal game model, where there is an additional condition that the output bits must satisfy, dramatically increases the computational complexity of its approximate quantum value: from polynomial-time to RE-complete.

In the standard XOR game model, Alice and Bob receive questions, $x \in X$ and $y \in Y$, they respond with bits, a and b, and they win if and only if $a \oplus b = f(x, y)$, for some predetermined function $f : X \times Y \to \{0, 1\}$. We define a **tilted XOR game** by making the winning condition slightly more stringent: for a distinguished $x_0 \in X$, the the winning condition is

$$\begin{cases} a \oplus b = f(x,y) \text{ and } a = 0 & \text{if } x = x_0 \\ a \oplus b = f(x,y) & \text{if } x \neq x_0. \end{cases}$$

In the context of classical strategies, the distinction between XOR games and tilted XOR games is inconsequential and they are NP-complete to approximate [J. Håstad. Some optimal inapproximability results. *J. of the ACM*, **48**(4):798–859, 2001]. With respect to quantum (entangled) strategies, it is known that the entangled value of an XOR game can be approximated to within exponentially fine precision in polynomial time. Our main result is that, for tilted XOR games, there exists a constant $\Delta > 0$ such that approximating the quantum value of a tilted XOR game to within precision Δ is RE-hard.

Since titled XOR games are a special case of **binary games** (where each party outputs one single bit), our result implies that binary games are RE-hard to approximate. Another way of viewing our result is as an MIP* protocol that enables an efficient verifier to determine whether or not a Turing machine halts, where each prover sends just one single bit to the verifier; the protocol has the property that there is a constant $\Delta > 0$ separating the completeness probability from the soundness probability.

Our result employs a gadget from the aforementioned paper of Håstad, that was used in the context of classical strategies (expressed in the language of approximation problems). We show that this same gadget has some remarkable properties that make it useful for the analysis of quantum strategies.

Contents

1 Introduction

In a nonlocal game (see Section 2 for formal definitions), two physically separated parties are given inputs (according to some fixed probability distribution) and, without any communication between them, they must produce outputs. There is fixed success-predicate that determines whether or not their answers for the questions that they were given constitute a win.

The classical value of a nonlocal game is the maximum success probability achievable by a strategy that is restricted to classical information; the quantum value is the maximum (or supremium) success probability attainable by strategies that can utilize pre-shared entanglement. It it known that, in general, computing an approximation of the classical value of a nonlocal game is an NP-complete problem (where the input-size is the total number of questions). And, in general, computing an approximation of the quantum value is RE-hard [JNV+20] (herein, we assume the tensor product model of entanglement).

Following the terminology in [CHTW04], there is a restricted class of nonlocal games, called **binary games**, where Alice and Bob's outputs are single bits. There is a further restricted class of nonlocal games, called **XOR games**, that are binary and also have the property that the success-predicate is a function of the XOR of the two output bits (in addition to the input values).

Approximating the classical value of an XOR game is much harder than approximating its quantum value. Specifically, results in [Hås01] imply that, for any $\varepsilon > 0$, distinguishing between success probability $\geq \frac{3}{4} - \varepsilon$ and $\leq \frac{11}{16} + \varepsilon$ is an NP-complete problem. On the other hand, XOR games have the special property that they can be formulated as semidefinite programs in a manner that permits them to be approximated in polynomial time. In fact, the precision can be exponentially high (meaning that the approximation error can be exponentially small, while polynomial-time is maintained).

How different are binary games from XOR games? There is one property of XOR games that carries over to binary games: in [CHTW04] it is shown that there is a perfect quantum strategy (that attains success probability 1) if and only if there is a perfect classical strategy.[1] This implies that there is a polynomial time algorithm for determining whether a binary game has a perfect quantum strategy. Our results imply that the computational complexity of approximating the quantum value of general binary games is extremely hard; namely, RE-hard, for some specific constant $\Delta > 0$. Note the sharp contrast with XOR games, that can be approximated to exponentially fine precision in polynomial time.

In fact, the binary games that we show to be RE-hard are only a slight variant of the definition an XOR game. If a and b are Alice and Bob's respective output bits for some inputs then the success-predicate for an XOR game is the condition that $a \oplus b = f(x, y)$, for some predetermined function $f : X \times Y \to \{0, 1\}$. For our variant of an XOR game—that we call a **tilted XOR game**—there is one distinguished input to Alice, x_0, such that the condition is

$$\begin{cases} a \oplus b = f(x,y) \text{ and } a = 0 & \text{if } x = x_0 \\ a \oplus b = f(x,y) & \text{if } x \neq x_0. \end{cases}$$

In fact, this seemingly minor tweak to definition of an XOR game is inconsequential in the context of classical strategies. However, our main result is that

> There exists a constant $\Delta > 0$ such that, for all $\varepsilon > 0$, it is RE-hard to distinguish between tilted XOR games having value $\geq \frac{3}{4} - \varepsilon$ vs. value $\leq \frac{3}{4} - \Delta$.

Our methodology builds on a recent result of Taller and Vidick [TV25], that shows E3-LIN games (generalizations of the Magic Square game to any set of linear equations, with three variabes per equation). In [Hås01], a reduction from E3-LIN games to XOR games is used to deduce the NP-hardness of approximating the classical values XOR games from the NP-hardness for the classical values of E3-LIN games (also shown in [Hås01]). We show that this same reduction can be used to deduce the RE-hardness of the quantum values of tilted XOR games from the RE-hardness of the quantum values of E3-LIN games shown in [TV25].

Our result employs a gadget used in [Hås01] for a classical reduction. We show that this same gadget has some remarkable properties that make it useful for the analysis of quantum strategies. Our technical analysis employs tools that characterize optimal and near-optimal protocols for XOR games in [Slo11].

2 Preliminaries

2.1 Notation

We use bold font to denote elements of a cartesian power: $\boldsymbol{x} = (x_1, \ldots, x_k) \in X^k$.

We consider only probability distributions π on finite sets X, so we identify them with functions $\pi : X \to [0, 1]$ such that $\sum_{x \in X} \pi(x) = 1$.

[1] An example of a game that has a perfect quantum strategy but does **not** have a perfect classical is the Magic Square game [Mer90, Per90].

Write $\|\cdot\|$ for the operator norm on $\mathcal{B}(H)$ and $\|\cdot\|_F$ for the Frobenius norm $\|T\|_F = \sqrt{\mathrm{Tr}(T^*T)}$. The operator absolute value is $|T| = \sqrt{T^*T}$. We denote the commutator of two operators as $[S,T] = ST - TS$ and the anticommutator as $\{S,T\} = ST + TS$.

A **positive operator-valued measurement (POVM)** on a Hilbert space H with a finite set of outcomes A is a set of positive semidefinite operators $\{P_a\}_{a\in A} \subseteq \mathcal{B}(H)$ such that $\sum_{a\in A} P_a = I$. A **projection-valued measurement (PVM)** is a POVM such that the P_a are projections.

A POVM is **binary** if $A = \{0,1\}$. The **observable** of a binary POVM $\{P_0, P_1\}$ is $P_0 - P_1$. Every hermitian operator A such that $-I \le A \le I$ is an observable, and it is the observable of a PVM if and only if it is unitary.

2.2 Nonlocal games

Definition 2.1. A **nonlocal game** is a tuple $G = (X, Y, A, B, \pi, V)$ where X, Y, A, B are finite sets, called Alice's questions, Bob's questions, Alice's answers, and Bob's answers, respectively; π is a probability distribution on $X \times Y$, called the question distribution; and $V : A \times B \times X \times Y \to \{0,1\}$, called the predicate.

A nonlocal game is **binary** if $A = B = \{0,1\}$.

Definition 2.2. A **strategy** for a nonlocal game G is a function $s : A \times B \times X \times Y \to [0,1]$ such that $s(\cdot,\cdot|x,y)$ is a probability distribution on $A \times B$ for all $x \in X$ and $y \in Y$. The **value** of a strategy s for G is

$$\omega(G,s) = \sum_{\substack{x\in X, y\in Y\\ a\in A, b\in B}} \pi(x,y) V(a,b|x,y) s(a,b|x,y).$$

A strategy s is

- **deterministic** if there exist functions $g : X \to A$ and $h : Y \to B$ such that $s(a,b|x,y) = \delta_{a,g(x)}\delta_{b,h(y)}$.
- **classical** if it belongs to the convex hull of the deterministic strategies.
- **quantum** if there exist finite-dimensional Hilbert spaces H_A and H_B, POVMs $\{P^x_a\}_{a\in A} \subseteq \mathcal{B}(H_A)$ for all $x \in X$ and $\{Q^y_b\}_{b\in B} \subseteq \mathcal{B}(H_B)$ for all $y \in Y$, and a state $|\psi\rangle \in \mathcal{B}(H_A \otimes H_B)$ such that $s(a,b|x,y) = \langle\psi|P^x_a \otimes Q^x_b|\psi\rangle$. Using Naimark's dilation theorem, the POVMs can always chosen to be PVMs.
- **tracial** if there exists a von Neumann algebra $\mathcal{M}$ with a normal tracial state τ, and POVMs $\{p^x_a\}_{a\in A} \subseteq \mathcal{M}$ for all $x \in X$ and $\{q^y_b\}_{b\in B} \subseteq \mathcal{M}$ for all $y \in Y$ such that $s(a,b|x,y) = \tau(p^x_a q^y_b)$. The finite-dimensional tracial strategies are exactly the quantum strategies where $|\psi\rangle$ is maximally entangled.

The **classical value** of G is the supremum over the values of all deterministic (or classical) strategies; it is denoted $\omega(G)$. The **quantum value** of G is the supremum over the values of all quantum strategies; it is denoted $\omega^*(G)$. We say a quantum strategy s is **optimal** if $\omega(G,s) = \omega^*(G)$, and **$\varepsilon$-optimal** if $\omega(G,s) \ge \omega^*(G) - \varepsilon$. We say a strategy is **perfect** if it is optimal and the optimal value is 1.

2.3 XOR games

Definition 2.3. An **XOR game** is a binary nonlocal game G for which there exists a function $f : X \times Y \to \{0,1\}$ such that $V(a,b|x,y) = \delta_{a+b,f(x,y)}$, where the addition is modulo 2.

The value of a strategy s for an XOR game G has a simple expression in terms of the observables A_x and B_y for the POVMs $\{P_a^x\}_a$ and $\{Q_b^y\}$, respectively:

$$\begin{aligned}\omega(G,s) &= \sum_{\substack{x\in X, y\in Y\\ a,b\in\{0,1\}:\, a+b=f(x,y)}} \pi(x,y) \langle\psi|P_a^x \otimes Q_b^y|\psi\rangle \\ &= \frac{1}{4}\sum_{x\in X,y\in Y}\pi(x,y)\sum_{a\in\{0,1\}}\langle\psi|(I+(-1)^a A_x)\otimes(I+(-1)^{a+f(x,y)}B_y)|\psi\rangle \\ &= \frac{1}{2}+\frac{1}{2}\sum_{x\in X,y\in Y}(-1)^{f(x,y)}\pi(x,y)\langle\psi|A_x\otimes B_y|\psi\rangle .\end{aligned}$$

This presentation implies that the XOR game can be fully characterised by the matrix H with entries $H_{x,y} = (-1)^{f(x,y)}\pi(x,y)$ and the strategy can be characterised by the **quantum correlation** $c(x,y) = \langle\psi|A_x \otimes B_y|\psi\rangle$. In general, a **correlation** is a function $c : X \times Y \to [-1,1]$ and the **bias** of a correlation is

$$\beta(G,c) = \sum_{x\in X,y\in Y} H_{x,y}c(x,y).$$

For a quantum strategy s with associated correlation c, $\beta(G,c) = 2\omega(G,s) - 1$. The **quantum bias** of G is $\beta^*(G) = 2\omega^*(G) - 1$.

Definition 2.4. A **vector correlation** c for an XOR game G is a function $c : X \times Y \to [-1,1]$ such that $c(x,y) = \langle u_x|v_y\rangle$ for some real unit vectors $|u_x\rangle, |v_y\rangle$.

The **vector bias** is the supremum over the biases of all vector correlation; it is denoted $\beta^{vect}(G)$. A vector correlation c is **optimal** if $\beta(G,c) = \beta^{vect}(G)$, and ε**-optimal** if $\beta(G,c) \geq \beta^{vect}(G) - 2\varepsilon$.

We include a factor of 2 in the definition of ε-optimal bias is there to ensure that the value is within ε of the optimal value.

The vector correlations are in bijective correspondence with quantum correlations, and therefore $\beta^{vect}(G) = \beta^*(G)$ [Tsi87]. This characterisation via vector correlations allows the bias of an XOR game to expressed as a semidefinite program (SDP). Following [CSUU08, Slo11], let $B = \frac{1}{2}\begin{bmatrix} 0 & H \\ H^T & 0\end{bmatrix}$. Then, the quantum bias of G is the optimal value of the SDP over $(|X|+|Y|)\times(|X|+|Y|)$-matrices M

$$\begin{aligned} &\text{maximise } \operatorname{Tr}(BM) \\ &\text{subject to } M_{i,i} = 1,\ \forall\, i \in X \sqcup Y \\ &\qquad\qquad M \geq 0. \end{aligned} \tag{1}$$

Here, M represents the Gram matrix of the vectors in the quantum strategy. The dual of this SDP is the SDP over vectors $\boldsymbol{a} \in \mathbb{R}^X$ and $\boldsymbol{b} \in \mathbb{R}^Y$

$$\begin{aligned} \text{minimise } & \frac{1}{2}\sum_{x\in X} a_x + \frac{1}{2}\sum_{y\in Y} b_y \\ \text{subject to } & \frac{1}{2}\Delta(\boldsymbol{a},\boldsymbol{b}) \geq B, \end{aligned} \tag{2}$$

where $\Delta(\boldsymbol{a},\boldsymbol{b})$ is the matrix with the entries of the vectors along the diagonal, and zeroes elsewhere. For an optimal strategy, the values a_x and b_y are called the **optimal row and column biases**, respectively, and satisfy

$$\sum_{y\in Y} H_{x,y} \langle u_x | v_y \rangle = a_x$$
$$\sum_{x\in X} H_{x,y} \langle u_x | v_y \rangle = b_y.$$

We make use of the following structure theorem for near-optimal XOR game strategies.

Theorem 2.5 ([Slo11] Theorem 3.1)**.** Let G be an XOR game, and let a_x for $x \in X$ and b_y for $y \in Y$ be the optimal row and column biases, respectively. Then, for any ε-optimal vector correlation $c(x,y) = \langle u_x | v_y \rangle$,

$$\Big\| \sum_{y\in Y} H_{x,y} |v_y\rangle - a_x |u_x\rangle \Big\|^2 = 10\sqrt{2(|X|+|Y|)\varepsilon},$$
$$\Big\| \sum_{x\in X} H_{x,y} |u_x\rangle - b_y |v_y\rangle \Big\|^2 = 10\sqrt{2(|X|+|Y|)\varepsilon}.$$

The computational complexity of XOR games is well understood.

Definition 2.6. Let $1 \geq c \geq s \geq 0$. E2-LIN$_{c,s}$ is the problem of deciding, for a given XOR game G, if $\omega(G) \geq c$ or $\omega(G) < s$, with the promise that one of these two holds.

E2-LIN$^*_{c,s}$ is the problem of deciding, for a given XOR game G, if $\omega^*(G) \geq c$ or $\omega^*(G) < s$, with the promise that one of these two holds.

The notation E2-LIN corresponds to the fact that XOR games are linear systems with two variables per equation.

For any constants $c > s$, E2-LIN$^*_{c,s} \in \mathsf{P}$ [CHTW04, Tsi87]. However, there exist $c > s$ such that E2-LIN$_{c,s}$ is NP-complete [Hås01]. We can choose $c = \frac{3}{4} - \varepsilon$ and $s = \frac{11}{16} + \varepsilon$ for any $\varepsilon > 0$ (see Section 2.5 for more details). Nevertheless, E2-LIN$_{1,s} \in \mathsf{P}$ for all s.

2.4 Linear constraint system games

Definition 2.7. A **3-linear constraint system (3-LCS)** consists of a set of variables X, and two sets of relations $R_0, R_1 \subseteq X^3$, corresponding to the linear equations $x + y + z = 0$ and $x + y + z = 1$ over $\{0, 1\}$.

Let $S = (X, R_0, R_1)$ be a 3-LCS and let π be a probability distribution on $\{0\} \times R_0 \cup \{1\} \times R_1$. Then, the **3-LCS game** of (S, π) is the nonlocal game $G(S,\pi) = (\{0\} \times R_0 \cup \{1\} \times R_1, X, \{0,1\}^3, \{0,1\}, \pi', V_S)$, where

$$\pi'((b,\boldsymbol{x}),x) = \begin{cases} \frac{\pi(b,\boldsymbol{x})}{3} & x = x_i \text{ for some } i \\ 0 & \text{else} \end{cases},$$

$$V_S(\boldsymbol{a}, a|(b,\boldsymbol{x}),x) = \begin{cases} 1 & a_1 + a_2 + a_3 = b \wedge (x = x_i \Rightarrow a = a_i) \\ 0 & \text{else.} \end{cases}$$

Perfect strategies for LCS games admit a useful characterisation [CLS17]. Suppose c is a perfect quantum strategy for $G(S,\pi)$, where we may without loss of generality assume that π is never 0. Let B_x for $x \in X$ be Bob's observables. Then, the shared state is maximally entangled and, for all $\boldsymbol{x} \in R_b$,

$$B_{x_1} B_{x_2} B_{x_3} = (-1)^b I,$$
$$[B_{x_i}, B_{x_j}] = 0.$$

In particular, this implies that all optimal strategies are tracial.

As for XOR games, LCS games induce a computational problem.

Definition 2.8. Let $1 \geq c \geq s \geq 0$. E3-LIN$_{c,s}$ is the problem of deciding, for a given 3-LCS game G, if $\omega(G) \geq c$ or $\omega(G) < s$, with the promise that one of these two holds.

E3-LIN$^*_{c,s}$ is the problem of deciding, for a given 3-LCS game G, if $\omega^*(G) \geq c$ or $\omega^*(G) < s$, with the promise that one of these two holds.

The classical case is similar to E2-LIN. E3-LIN$_{1,s} \in \mathsf{P}$ for all s, but there exist $c > s$ such that E3-LIN$_{c,s}$ is NP-complete. Furthermore, unlike in the E2-LIN case, c may be chosen to be arbitrarily close to 1; in fact we may take $c = 1 - \varepsilon$ and $s = \frac{5}{6} + \varepsilon$ for any $\varepsilon > 0$ [Hås01].

However, for the quantum case, the complexity increases drastically. E3-LIN$^*_{1,1}$ is undecidable [Slo19] and there exist $c > s$ such that E3-LIN$^*_{c,s}$ is RE-complete [TV25]. In latter case, c may be chosen to be arbitrarily close to 1: in fact against tracial quantum strategies, we may choose $c = 1 - \varepsilon$ and $s = \frac{35}{36} - \varepsilon$ for any ε. However, extending to general quantum strategies the lowest value for s we can attain is $s = 1 - \frac{1}{3 \cdot (5328)^4} + \varepsilon \approx 1 - 4 \times 10^{-16}$ [Cul26]. The complexity of E3-LIN$^*_{1,s}$ for $s < 1$ is still not well-understood.

2.5 Håstad's E3-Lin to E2-Lin reduction

The NP-completeness of E2-LIN is ascertained in [Hås01] via a gadget reduction from E3-LIN. The reduction relies on the following game.

Definition 2.9 (Håstad's cube game)**.** Let $X_0, X_1 \subseteq \{0,1\}^3$ be the sets of bit strings of even and odd parity, respectively. Then, the **bipartite cube game** is the XOR game $G^\square$ with question sets $X = X_0$ and $Y = X_1$, and game matrix $H_{x,y} = \frac{1}{16}$ for $y = \neg x$ and $H_{x,y} = -\frac{1}{16}$ otherwise.

In [Hås01], this game was originally phrased as a system of equations, but it is natural to represent it as an XOR game.

Lemma 2.10 ([Hås01])**.** The value $\omega(G^{\square}) = \frac{3}{4}$. If $c(a,b|x,y) = \delta_{a,g(x)}\delta_{b,h(y)}$ is an optimal deterministic strategy, then $g(011)+g(101)+g(110) = g(000)$. If a deterministic strategy is not optimal, then its value is $\leq \frac{5}{8}$.

To finish this section, we sketch the reduction from E3-LIN to E2-LIN due to [Hås01]. Let (S,π) be an instance of E3-LIN. We construct an instance of E2-LIN as follows. We replace each equation by a copy of the bipartite cube game, identifying the variables x_1, x_2, and x_3 in the equation with 011, 101, and 110 in the cube, respectively. We also identify all the variables corresponding to 000 between the cubes, and for each cube corresponding to an equation of the form $x_1 + x_2 + x_3 = 1$, we flip the sign of the relations involving 000 — this is equivalent to flipping the parity of the variable 000 in those cubes. Since the value of an assignment to an E2-LIN instance is invariant under negation, we may assume that, for any classical strategy the variable 000 is assigned value 0.

Now, if (S,π) is a yes instance, there exists a classical assignment $1 - O(\varepsilon)$ of the equations are satisfied, which implies that $1 - O(\varepsilon)$ of the cubes may be won with probability $\frac{3}{4}$. Hence the value of the new game is $\geq \frac{3}{4} - O(\varepsilon)$. Now, suppose (S,π) is is an instance of E3-LIN such that the E2-LIN instance constructed as above has value $\geq \frac{11}{16} + \varepsilon$ — to ascertain soundness of the reduction, it suffices to see that the classical value of $G(S,\pi)$ is $\geq \frac{5}{6} + O(\varepsilon)$. Since the value of the cube when the corresponding 3-linear equation is not satisfied is at most $\frac{10}{16}$, the lower bound in the winning probability implies that at least $\frac{1}{2} + O(\varepsilon)$ of the equations are satisfied. For a satisfied equation, the value of the corresponding term in the LCS game is 1, and for an unsatisfied equation, we can choose a strategy where, by making Alice answer consistently on 2 of the 3 variables, the value is $\frac{2}{3}$. Putting these together, we find a strategy for the original E3-LIN instance with value $\frac{5}{6} + O(\varepsilon)$.

3 Tilted XOR games

In this section, we introduce the class of binary games for which we show hardness of approximation.

Definition 3.1. A **tilted XOR game** is a nonlocal game $G = (X, Y, A, B, \pi, V)$ where $A = B = \{0,1\}$, X, Y (may) each contain a distinguished element $\perp$, and there exists a function $f : X \times Y \to \{0,1\}$ such that

$$V(a,b|x,y) = \begin{cases} 1 & x = y = \perp \\ \delta_{a,f(x,\perp)} & y = \perp \\ \delta_{b,f(\perp,y)} & x = \perp \\ \delta_{a+b,f(x,y)} & \text{else.} \end{cases}$$

This corresponds to an XOR game with additional questions where only either Alice or Bob is asked for a bit.

We say G is **one-sided** if either $\perp \notin X$ or $\perp \notin Y$, *i.e.* only one of Alice or Bob is asked the distinguished question.

We can express the value of a quantum strategy $s(a,b|x,y) = \langle\psi|P^x_a \otimes Q^y_b|\psi\rangle$ for G using the associated quantum correlation $c(x,y) = \langle\psi|A_x \otimes B_y|\psi\rangle$. Without loss of generality, we may suppose that $A_\perp = I$ and $B_\perp = I$, since Alice and Bob's answers on the distinguished question do not affect the value. Then, the value of s is

$$\begin{aligned}
\omega(G,s) = \pi(\perp,\perp) &+ \sum_{x\in X\setminus\{\perp\}} \pi(x,\perp)\,\langle\psi|P^x_{f(x,\perp)} \otimes I|\psi\rangle + \sum_{y\in Y\setminus\{\perp\}} \pi(\perp,y)\,\langle\psi|I \otimes Q^x_{f(\perp,y)}|\psi\rangle \\
&+ \sum_{\substack{x\in X\setminus\{\perp\},y\in Y\setminus\{\perp\}\\ a,b\in\{0,1\}:\,a+b=f(x,y)}} \pi(x,y)\,\langle\psi|P^x_a \otimes Q^y_b|\psi\rangle \\
&= \pi(\perp,\perp) + \frac{1}{2}\sum_{x\in X\setminus\{\perp\}} \pi(x,\perp)\,\langle\psi|(1+(-1)^{f(x,\perp)}A_x)\otimes B_\perp|\psi\rangle \\
&\quad+ \frac{1}{2}\sum_{y\in Y\setminus\{\perp\}} \pi(\perp,y)\,\langle\psi|A_\perp \otimes (I+(-1)^{f(\perp,y)}B_y)|\psi\rangle \\
&\quad+ \frac{1}{4}\sum_{x\in X\setminus\{\perp\},y\in Y\setminus\{\perp\}} \pi(x,y) \sum_{a\in\{0,1\}} \langle\psi|(I+(-1)^a A_x)\otimes(I+(-1)^{a+f(x,y)}B_y)|\psi\rangle \\
&= \frac{1}{2} + \frac{1}{2}\sum_{x\in X,y\in Y} (-1)^{f(x,y)}\pi(x,y)\,\langle\psi|A_x \otimes B_y|\psi\rangle\,.
\end{aligned}$$

As for an XOR game, let the game matrix $H_{x,y} = (-1)^{f(x,y)}\pi(x,y)$ and the bias of a correlation $\beta(G,c) = \sum_{x\in X,y\in Y} H_{x,y}c(x,y)$.

Example 3.2. Common examples of tilted XOR games are the tilted CHSH game [AMP12] and the I3322 game [Fro81, Śli03, CG04]. The former is one-sided but the latter is not. Note that these games are usually presented via their Bell inequality presentations, amounting to rescalings of the bias.

Definition 3.3. The **XOR relaxation** of a tilted XOR game G is the XOR game G_{XOR} with the same distribution where $V_{XOR}(a,b|x,y) = \delta_{a+b,f(x,y)}$, *i.e.* the distinguished questions $\perp$ are treated as usual questions in an XOR game.

Lemma 3.4. Let G be a tilted XOR game. Then, $\omega(G_{XOR}) \geq \omega(G)$ and $\omega^*(G_{XOR}) \geq \omega^*(G)$. If G is one-sided, then $\omega(G) = \omega(G_{XOR})$.

Proof. For the first part, note that via the representation of the tilted XOR game bias in terms of correlations, every strategy for G corresponds to a strategy for G_{XOR} where Alice and Bob always output 0 on question $\perp$. Thus the classical strategies for G form a subset of the classical strategies for G_{XOR}, and the quantum strategies for G form a subset of the quantum strategies for G_{XOR}, giving the upper bounds.

For the second part, let G be a tilted XOR game such that $\perp\notin Y$, and let $s(a,b|x,y) = \delta_{g(x),a}\delta_{h(y),b}$ be an optimal classical strategy for G_{XOR}. Then, define the classical strategy s' for G via the functions $g'(x) = g(x)+g(\perp)$ and $h'(y) = h(y)+g(\perp)$. Since $g'(x)+h'(y) = g(x)+h(y)$ and $g'(\perp) = 0$, this is a strategy for G with the same value as the value of s for G_{XOR}. So, the optimal classical value of G upper bounds the optimal classical value of G_{XOR}. The case where $\perp\notin X$ is symmetric. ■

In the same way as XOR games correspond to linear systems with two variables per equation, tilted XOR games correspond to linear systems with one or two variables per equation. This induces a computational problem.

Definition 3.5. Let $1 \geq c \geq s \geq 0$. E1-or-2-LIN$_{c,s}$ is the problem of deciding, for a given tilted XOR game G, if $\omega(G) \geq c$ or $\omega(G) < s$, with the promise that one of these two holds.

E1-or-2-LIN$^*_{c,s}$ is the problem of deciding, for a given tilted XOR game G, if $\omega^*(G) \geq c$ or $\omega^*(G) < s$, with the promise that one of these two holds.

We also define One-Sided-E1-or-2-LIN$_{c,s}$ and One-Sided-E1-or-2-LIN$^*_{c,s}$ by restricting to one-sided tilted XOR games.

It follows from the hardness of E2-LIN [Hås01] and Definition 3.4 that E1-or-2-LIN$_{c,s}$ is NP-complete, with $c = \frac{3}{4} - \varepsilon$ and $s = \frac{11}{16} + \varepsilon$. Further, it is clear that One-Sided-E1-or-2-LIN$_{c,s}$ and One-Sided-E1-or-2-LIN$^*_{c,s}$ trivially reduce to E1-or-2-LIN$_{c,s}$ and E1-or-2-LIN$^*_{c,s}$, respectively. Also, for $c > s$, E1-or-2-LIN$^*_{c,s} \in$ RE as the problem consists of approximating the value of a nonlocal game to constant precision.

4 Reduction from E3-Lin to tilted XOR games

In this section, we prove the main result of this work.

Theorem 4.1. There exist $c > s$ such that One-Sided-E1-or-2-LIN$^*_{c,s}$ is RE-complete.

From the proof, we may choose explicit values as $c = \frac{3}{4} - \varepsilon$ for any $\varepsilon > 0$, and $s = \frac{3}{4} - 10^{-24}$. The theorem also demonstrates RE-completeness of E1-or-2-LIN$^*_{c,s}$ with the same parameters. We prove the result via a reduction from E3-LIN*, in an analogous way to [Hås01].

4.1 Improved XOR game structure theorem

In this section, we prove a tighter average-case version of Definition 2.5 [Slo11, Theorem 3.1] when there is a symmetry condition on the XOR game.

Lemma 4.2. Let G be an XOR game such that $|X| = |Y|$, and suppose that all the optimal row and column biases are equal $a_x = b_y =: \alpha$. Then, for any ε-optimal vector correlation $c(x, y) = \langle u_x | v_y \rangle$,

$$\sum_{x \in X} \Big\| \sum_{y \in Y} H_{x,y} |v_y\rangle - \alpha |u_x\rangle \Big\|^2 \leq 8(\alpha + 2\beta^*(G))\varepsilon$$

By symmetry, we also have that

$$\sum_{y \in Y} \Big\| \sum_{x \in X} H_{x,y} |u_x\rangle - \alpha |v_y\rangle \Big\|^2 \leq 8(\alpha + 2\beta^*(G))\varepsilon$$

Proof. First, write $n = |X|$ and $\beta = \beta^*(G)$; we know $\beta = n\alpha$. By definition, there exists some $\varepsilon' \leq 2\varepsilon$ such that $\sum_{x,y} H_{x,y} \langle u_x|v_y\rangle = \beta - \varepsilon'$. Then, we expand the left-hand side from the lemma statement to get that

$$\begin{aligned}\sum_x \Big\| \sum_y H_{x,y}|v_y\rangle - \alpha|u_x\rangle \Big\|^2 &= \sum_x \Big(\alpha^2 - 2\alpha \sum_y H_{x,y} \langle u_x|v_y\rangle + \Big\| \sum_y H_{x,y}|v_y\rangle \Big\|^2 \Big) \\ &= -\alpha\beta + 2\alpha\varepsilon' + \sum_x \Big\| \sum_y H_{x,y}|v_y\rangle \Big\|^2\end{aligned}$$

Now, we focus on bounding the last term. Following the argument of [Slo11, Theorem 3.1], let $|u'_x\rangle$ be the normalisation of $\sum_y H_{x,y}|v_y\rangle$ (if $\sum_y H_{x,y}|v_y\rangle = 0$, we take $|u'_x\rangle$ to be an arbitrary unit vector). Since

$$\sum_{x,y} H_{x,y} \langle u'_x|v_y\rangle = \sum_x \Big\| \sum_y H_{x,y}|v_y\rangle \Big\| \geq \sum_{x,y} H_{x,y} \langle u_x|v_y\rangle ,$$

the vector correlation $c'(x,y) = \langle u'_x|v_y\rangle$ is an ε-optimal strategy as well, with bias $\beta - \varepsilon'_B$ for some $\varepsilon'_B \leq \varepsilon' \leq 2\varepsilon$. Then,

$$\begin{aligned}\sum_x \Big\| \sum_y H_{x,y}|v_y\rangle \Big\|^2 &= \sum_x \Big\| \sum_y H_{x,y}|v_y\rangle - \alpha|u'_x\rangle - \alpha|u'_x\rangle \Big\|^2 \\ &= \sum_x \Big(\alpha^2 - 2\alpha \sum_y H_{x,y} \langle u'_x|v_y\rangle + 2\alpha^2 + \Big\| \sum_y H_{x,y}|v_y\rangle - \alpha|u'_x\rangle \Big\|^2 \Big) \\ &= \alpha\beta + 2\alpha\varepsilon'_B + \sum_x \Big(\sum_y H_{x,y} \langle u'_x|v_y\rangle - \alpha \Big)^2.\end{aligned}$$

As in the SDP formulation, write M for the $2n \times 2n$ Gram matrix of the vectors $|u'_x\rangle$ and $|v_y\rangle$ over x and y, and let

$$S = \frac{1}{2} \begin{bmatrix} \alpha I & -H \\ -H^T & \alpha I \end{bmatrix}.$$

We have that $\varepsilon'_B = \mathrm{Tr}(SM)$, $\alpha I \geq S \geq 0$, and $\|M\| \leq \mathrm{Tr}(M) \leq 2n$. Then,

$$\begin{aligned}\sum_x \Big(\sum_y H_{x,y} \langle u'_x|v_y\rangle - \alpha \Big)^2 &= 4 \sum_{x \in X} (SM)^2_{x,x} \\ &\leq 4\|SM\|_F^2 = 4\,\mathrm{Tr}(SM^2S) \\ &\leq 4\|S\|\|M\|\,\mathrm{Tr}(SM) \\ &\leq 4(\alpha)(2n)\varepsilon'_B = 8\beta\varepsilon'_B.\end{aligned}$$

Putting it all together,

$$\begin{aligned}&\sum_x \Big\| \sum_y H_{x,y}|v_y\rangle - \alpha|u_x\rangle \Big\|^2 \leq -\alpha\beta + 2\alpha\varepsilon' + (\alpha\beta + 2\alpha\varepsilon'_B + 8\beta\varepsilon'_B) \\ &\leq 8(\alpha + 2\beta)\varepsilon.\end{aligned}$$

■

4.2 The tilted bipartite cube game

In this section, we study the structure of a tilted XOR arising from the bipartite cube game, which we will make use of as a gadget to reduce from E3-Lin.

Definition 4.3. The **tilted bipartite cube game** is the tilted XOR game $G^\Diamond$ with the same data as $G^\Box$ from Definition 2.9 and distinguished variable 000.

It is clear from the definition that $G^\Diamond_{XOR} = G^\Box$. Also, since the distinguished question is only sent to Alice, it follows by Definition 3.4 that $\omega(G^\Diamond) = \omega(G^\Box) = \frac{3}{4}$. Now, we bound the quantum value of $G^\Diamond$.

Lemma 4.4. The quantum value $\omega^*(G^\Box) = \frac{3}{4}$.

Since $\omega^*(G^\Diamond)$ is sandwiched in between $\omega^*(G^\Box)$ and $\omega(G^\Diamond) = \frac{3}{4}$, this implies $\omega^*(G^\Diamond) = \frac{3}{4}$ as well.

Proof. It is clear that $\omega^*(G^\Box) \geq \omega(G^\Box) = \frac{3}{4}$. We use the dual SDP (2) discussed in Section 2.3 to upper bound the quantum value of $G^\Box$. Ordering the elements of X_0 as $(000, 011, 101, 110)$ and the elements of X_1 as $(111, 100, 010, 001)$ the game matrix is

$$H = \frac{1}{16}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

Taking the square, we find that

$$H^2 = \frac{1}{16^2}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \frac{1}{64}I,$$

which implies that the eigenvalues of H are $\pm\frac{1}{8}$. Then, the eigenvalues of $B = \frac{1}{2}\left[\begin{smallmatrix} 0 & H \\ H^T & 0 \end{smallmatrix}\right]$ are $\pm\frac{1}{16}$. Therefore, $\frac{1}{16}I \geq B$, giving that $a_x = b_y = \frac{1}{8}$ is a feasible point for the dual SDP (2) for $G^\Box$, giving an upper bound on the bias $\beta^*(G^\Box) \leq \frac{1}{2}\sum_{x\in X_0} a_x + \frac{1}{2}\sum_{y\in X_1} b_y = \frac{1}{2}$. This gives the upper bound on the value $\omega^*(G^\Box) \leq \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}$. ■

It can be seen from the proof above that the marginal row and column biases are all $\alpha = \frac{1}{8}$. Hence, Definition 4.2 applies to $G^\Box$. Now, we show that winning $G^\Diamond$ near-optimally implies that Alice's observables approximately satisfy a linear constraint. First, as a warmup, we consider the case of an optimal strategy.

Lemma 4.5. Let $c(x, y) = \langle\psi|A_x \otimes B_y|\psi\rangle$ be an optimal correlation for $G^\Diamond$. Suppose that the observables are unitaries and the state $|\psi\rangle$ has full support on Alice's system. Then, A_{011}, A_{101}, and A_{110} commute and $A_{011}A_{101}A_{110} = I$.

Proof. Since $\omega^*(G^\square) = \omega^*(G^\Diamond)$, c is an optimal correlation for $G^\square$. Let $|u_x\rangle = (A_x \otimes I)|\psi\rangle$ and $|v_y\rangle = (I \otimes B_y)|\psi\rangle$. Since the optimal value and the Frobenius norm are equal in terms of the usual and the real inner product, we have by Definition 4.2 that $\sum_{x\in X} H_{x,y}|u_x\rangle = \frac{1}{8}|v_y\rangle$, and therefore

$$\sum_x H_{x,y}(A_x \otimes I)|\psi\rangle = \frac{1}{8}(I \otimes B_y)|\psi\rangle.$$

Taking the square,

$$\frac{1}{64}|\psi\rangle = \frac{1}{64}(I \otimes B_y)^2|\psi\rangle = \frac{1}{8}\sum_x H_{x,y}(A_x \otimes B_y)|\psi\rangle = \Big(\sum_x H_{x,y}(A_x \otimes I)\Big)^2|\psi\rangle.$$

Since $|\psi\rangle$ has full support on Alice's system, $\left(\sum_x H_{x,y}A_x\right)^2 = \frac{1}{64}I$. Writing e_i for the canonical basis vectors in $\{0,1\}^3$, we have for any $y \in \{0,1\}^3$ of odd parity that

$$(A_{\neg y} - A_{y+e_1} - A_{y+e_2} - A_{y+e_3})^2 = 4I,$$

as $|H_{x,y}| = \frac{1}{16}$. Since c is a strategy for the tilted XOR game, we know that $A_{000} = I$. Writing $A_1 = A_{011}$, $A_2 = A_{101}$, and $A_3 = A_{110}$, we get the four relations

$$\begin{aligned}
(I - A_1 - A_2 - A_3)^2 &= 4I,\\
(-I + A_1 - A_2 - A_3)^2 &= 4I,\\
(-I - A_1 + A_2 - A_3)^2 &= 4I,\\
(-I - A_1 - A_2 + A_3)^2 &= 4I.
\end{aligned}$$

Adding the first two relations gives

$$\begin{aligned}
8I &= ((I - A_1) - (A_2 + A_3))^2 + (-(I - A_1) - (A_2 + A_3))^2\\
&= 2\Big((I - A_1)^2 + (A_2 + A_3)^2\Big)\\
&= 2(I - 2A_1 + I + I + A_2A_3 + A_3A_2 + I)\\
&= 2(4I - 2A_1 + \{A_2, A_3\}),
\end{aligned}$$

which can be rearranged to give $\{A_2, A_3\} = 2A_1$. By symmetry of the relations, we get the other anticommutation relations $\{A_1, A_2\} = 2A_3$ and $\{A_1, A_3\} = 2A_2$. Consider the following:

$$\begin{aligned}
4A_1 &= \{A_2, 2A_3\}\\
&= \{A_2, \{A_1, A_2\}\}\\
&= A_2(A_1A_2 + A_2A_1) + (A_1A_2 + A_2A_1)A_2\\
&= 2(A_2A_1A_2 + A_1).
\end{aligned}$$

Rearranging, we find $A_2A_1A_2 - A_1 = 0$, and then multiplying by A_2 gives the commutation relation $[A_1, A_2] = 0$. In the same way, we find the other commutation relations $[A_1, A_3] = [A_2, A_3] = 0$. Hence, Alice's observables commute. To finish the proof, we see using the commutation relations that

$$A_1A_2A_3 = \frac{1}{2}A_1\{A_2, A_3\} = A_1^2 = I.$$

■

Now, we can prove the robust version of the above result.

Proposition 4.6. Let $c(x,y) = \langle\psi|A_x \otimes B_y|\psi\rangle$ be an ε-optimal quantum correlation for $G^\diamond$ such that the observables are unitaries. Then,

$$\left\|\Big((A_{011}A_{101}A_{110} - I)\otimes I\Big)|\psi\rangle\right\| \le 188\sqrt{13\varepsilon}.$$

By symmetry, the same upper bound holds independent of the order of the observables.

Proof. Let $A_x = P_0^x - P_1^x$ and $B_y = Q_0^y - Q_1^y$ be Alice and Bob's observables, respectively, where $A_{000} = I$. Then, as $\alpha = \frac{1}{8}$ and $\beta^*(G^\diamond) = \frac{3}{4}$, Definition 4.2 implies that

$$\sum_y \left\|\sum_x H_{x,y}(A_x \otimes I)|\psi\rangle - \frac{1}{8}(I\otimes B_y)|\psi\rangle\right\|^2 \le 8\Big(\frac{1}{8} + 2\frac{3}{4}\Big)\varepsilon = 13\varepsilon. \tag{3}$$

Now, let $|\psi\rangle = \sum_i \sqrt{p_i}|a_i\rangle \otimes |b_i\rangle$ be the Schmidt decomposition of $|\psi\rangle$, and take $\lambda : H_B \to H_A$ to be the operator $\lambda = \sum_i \sqrt{p_i}\,|a_i\rangle\langle b_i|$. Then, $\lambda\lambda^* = \psi_A$ and $\lambda^*\lambda = \psi_B$, the marginals of $|\psi\rangle$ on H_A and H_B, respectively. We can rewrite (3) in terms of the Frobenius norm as

$$\sum_y \left\|\sum_x H_{x,y}A_x\lambda - \frac{1}{8}\lambda\overline{B}_y\right\|_F^2 \le 13\varepsilon,$$

where the complex conjugate is with respect to the basis $\{|b_i\rangle\}_i$. Writing $h_{x,y} = 16H_{x,y}$, we have $|h_{x,y}| = 1$ and

$$\sum_y \left\|\sum_x h_{x,y}A_x\lambda - 2\lambda\overline{B}_y\right\|_F^2 \le 2^8 \cdot 13\varepsilon.$$

Next, we remove the dependence on $\overline{B}_y$:

$$\begin{aligned}
\left\|\Big(\sum_x h_{x,y}A_x\Big)^2\lambda - 4\lambda\right\|_F &\le \left\|\Big(\sum_x h_{x,y}A_x\Big)^2\lambda - 2\sum_x h_{x,y}A_x\lambda\overline{B}_y\right\|_F + \left\|2\sum_x h_{x,y}A_x\lambda\overline{B}_y - 4\lambda\right\|_F \\
&\le \left\|\sum_x h_{x,y}A_x\right\|\left\|\sum_x h_{x,y}A_x\lambda - 2\lambda\overline{B}_y\right\|_F + 2\left\|\sum_x h_{x,y}A_x\lambda - 2\lambda\overline{B}_y\right\|_F\left\|\overline{B}_y\right\| \\
&\le 6\left\|\sum_x h_{x,y}A_x\lambda - 2\lambda\overline{B}_y\right\|_F,
\end{aligned}$$

and therefore

$$\sum_y \left\|\Big(\sum_x h_{x,y}A_x\Big)^2\lambda - 4\lambda\right\|_F^2 \le 36\sum_y\left\|\sum_x h_{x,y}A_x\lambda - 2\lambda\overline{B}_y\right\|_F^2 \le 2^{10}\cdot 117\varepsilon.$$

Now, as in the proof of Definition 4.5, let $A_1 = A_{011}$, $A_2 = A_{101}$, and $A_3 = A_{110}$. Then we expand to see that

$$\begin{aligned}
&\left\|(I - A_1 - A_2 - A_3)^2\lambda - 4\lambda\right\|_F^2 + \left\|(-I + A_1 - A_2 - A_3)^2\lambda - 4\lambda\right\|_F^2 \\
&\quad + \left\|(-I - A_1 + A_2 - A_3)^2\lambda - 4\lambda\right\|_F^2 + \left\|(-I - A_1 - A_2 + A_3)^2\lambda - 4\lambda\right\|_F^2 \le 2^{10}\cdot 117\varepsilon.
\end{aligned}$$

Using the facts that

$$(I - A_1 - A_2 - A_3)^2 + (-I + A_1 - A_2 - A_3)^2 = 2(4I - 2A_1 + \{A_2, A_3\})$$
$$(-I - A_1 + A_2 - A_3)^2 + (-I - A_1 - A_2 + A_3)^2 = 2(4I + 2A_1 - \{A_2, A_3\}),$$

we find that

$$\begin{aligned}
\|\{A_2, A_3\}\lambda - 2A_1\lambda\|_F^2 &= \frac{1}{8}\Big\|(I - A_1 - A_2 - A_3)^2\lambda + (-I + A_1 - A_2 - A_3)^2\lambda - 8\lambda\Big\|_F^2 \\
&\quad + \frac{1}{8}\Big\|(-I - A_1 + A_2 - A_3)^2\lambda + (-I - A_1 - A_2 + A_3)^2\lambda - 8\lambda\Big\|_F^2 \\
&\leq \frac{1}{4}\Big\|(I - A_1 - A_2 - A_3)^2\lambda - 4\lambda\Big\|_F^2 + \frac{1}{4}\Big\|(-I + A_1 - A_2 - A_3)^2\lambda - 4\lambda\Big\|_F^2 \\
&\quad + \frac{1}{4}\Big\|(-I - A_1 + A_2 - A_3)^2\lambda - 4\lambda\Big\|_F^2 + \frac{1}{4}\Big\|(-I - A_1 - A_2 + A_3)^2\lambda - 4\lambda\Big\|_F^2 \\
&\leq 2^8 \cdot 117\varepsilon.
\end{aligned}$$

By symmetry, the other anticommutation relations hold as well. Next, let $\widetilde{B}_1 = \frac{1}{2}\Big(\overline{B}_{100} - \overline{B}_{010} - \overline{B}_{001} - \overline{B}_{111}\Big)$, $\widetilde{B}_2 = \frac{1}{2}\Big(\overline{B}_{010} - \overline{B}_{100} - \overline{B}_{111} - \overline{B}_{001}\Big)$, and $\widetilde{B}_3 = \frac{1}{2}\Big(\overline{B}_{001} - \overline{B}_{111} - \overline{B}_{100} - \overline{B}_{010}\Big)$. By construction, $\|\widetilde{B}_i\| \leq 2$ and by Definition 4.2, $\sum_i \|A_i\lambda - \lambda\widetilde{B}_i\|_F^2 \leq 2^6 \cdot 13\varepsilon$. Now, note that

$$\begin{aligned}
A_1A_2A_3 - I &= \frac{1}{2}(A_1(\{A_2, A_3\} - 2A_1) + A_1[A_2, A_3]) \\
&= \frac{1}{2}A_1(\{A_2, A_3\} - 2A_1) - \frac{1}{4}A_1A_2(\{A_2, \{A_2, A_3\}\} - 4A_3) \\
&= \frac{1}{4}A_1(\{A_2, A_3\} - 2A_1) - \frac{1}{2}A_1A_2(\{A_2, A_1\} - 2A_3) - \frac{1}{4}A_1A_2(\{A_2, A_3\} - 2A_1)A_2
\end{aligned}$$

Then, we can use this to bound

$$\begin{aligned}
\|(A_1A_2A_3 - I)\lambda\|_F &\leq \frac{1}{4}\|A_1(\{A_2, A_3\} - 2A_1)\lambda\|_F + \frac{1}{2}\|A_1A_2(\{A_2, A_1\} - 2A_3)\lambda\|_F \\
&\quad + \frac{1}{4}\|A_1A_2(\{A_2, A_3\} - 2A_1)A_2\lambda\|_F \\
&\leq \frac{1}{4}\|(\{A_2, A_3\} - 2A_1)\lambda\|_F + \frac{1}{2}\|(\{A_2, A_1\} - 2A_3)\lambda\|_F \\
&\quad + \frac{1}{4}\Big\|(\{A_2, A_3\} - 2A_1)\lambda\widetilde{B}_2\Big\|_F + \frac{1}{4}\Big\|(\{A_2, A_3\} - 2A_1)(A_2\lambda - \lambda\widetilde{B}_2)\Big\|_F \\
&\leq \frac{3}{4}\|(\{A_2, A_3\} - 2A_1)\lambda\|_F + \frac{1}{2}\|(\{A_2, A_1\} - 2A_3)\lambda\|_F + \Big\|A_2\lambda - \lambda\widetilde{B}_2\Big\|_F \\
&\leq (\frac{3}{4} + \frac{1}{2})2^4\sqrt{117\varepsilon} + 2^3\sqrt{13\varepsilon} \\
&= 188\sqrt{13\varepsilon}.
\end{aligned}$$

■

4.3 Proof of the reduction

First, we show that near-optimal strategies for the tilted bipartite cube game give rise to operators that are near-perfect for a single E3-LIN relation. This is necessary in the proof of soundness.

Lemma 4.7. Let $s(a,b|x,y) = \langle\psi|P_a^x \otimes Q_b^y|\psi\rangle$ be an ε-optimal strategy for $G^\Diamond$ such that the players' measurements are PVMs. Then, there exists a state $|\phi\rangle \in H_A \otimes H_A$ that depends only on $|\psi\rangle$, and a POVM $\{\Pi_{\boldsymbol{a}}\}_{\boldsymbol{a}\in\{0,1\}^3} \subseteq \mathcal{B}(H_A)$ such that

$$\frac{1}{3}\sum_{\substack{\boldsymbol{a}\in\{0,1\}^3\\ a_1+a_2+a_3=0}} \langle\phi|\Pi_{\boldsymbol{a}} \otimes (P_{a_1}^{011} + P_{a_2}^{101} + P_{a_3}^{110})|\phi\rangle \geq 1 - C\varepsilon,$$

where $C = 1302579$.

Proof. We use the same notation as in Definition 4.6. Fix an orthonormal basis $\{|k\rangle\}_k$ of H_A, and let $|\phi\rangle = \sum_k |k\rangle \otimes \psi_A^{1/2}|k\rangle$. Let $\Pi_{\boldsymbol{a}} = \overline{P_{a_3}^3 P_{a_2}^2 P_{a_1}^1 P_{a_2}^2 P_{a_3}^3}$. Write

$$w = \frac{1}{3}\sum_{\substack{\boldsymbol{a}\in\{0,1\}^3\\ a_1+a_2+a_3=0}} \langle\phi|\Pi_{\boldsymbol{a}} \otimes (P_{a_1}^1 + P_{a_2}^2 + P_{a_3}^3)|\phi\rangle\,.$$

We will lower bound this by upper bounding

$$1 - w = \frac{1}{3}\sum_{\substack{\boldsymbol{a}\in\{0,1\}^3\\ a_1+a_2+a_3=0}} \langle\phi|\Pi_{\boldsymbol{a}} \otimes (P_{\neg a_1}^1 + P_{\neg a_2}^2 + P_{\neg a_3}^3)|\phi\rangle + \sum_{\substack{\boldsymbol{a}\in\{0,1\}^3\\ a_1+a_2+a_3=1}} \langle\phi|\Pi_{\boldsymbol{a}} \otimes I|\phi\rangle\,.$$

First, we want to show that A_i approximately commutes with $\psi_A^{1/2}$. Due to Definition 4.2, $\|A_i\lambda - \lambda\widetilde{B}_i\|_F \leq 8\sqrt{13\varepsilon}$. with the goal of removing $\widetilde{B}_i$, we first round it to an order-2 unitary. We have

$$\|\lambda(I - \widetilde{B}_i^2)\|_F \leq \|A_i^2\lambda - A_i\lambda\widetilde{B}_i\|_F + \|A_i\lambda\widetilde{B}_i - \lambda\widetilde{B}_i^2\|_F \leq 3\|A_i\lambda - \lambda\widetilde{B}_i\|_F \leq 24\sqrt{13\varepsilon}.$$

Now, let $C_i = \operatorname{sgn}(\widetilde{B}_i)$, which is an order-2 unitary by construction. We have that

$$(I - \widetilde{B}_i^2)^2 = (C_i - \widetilde{B}_i)^2(C_i + \widetilde{B}_i)^2 \geq (C_i - \widetilde{B}_i)^2,$$

so $\|\lambda(C_i - \widetilde{B}_i)\|_F \leq \|\lambda(I - \widetilde{B}_i^2)\|_F \leq 24\sqrt{13\varepsilon}$. Therefore, we can replace $\widetilde{B}_i$ with C_i and find

$$\|A_i\lambda - \lambda C_i\|_F \leq \|A_i\lambda - \lambda\widetilde{B}_i\|_F + \|\lambda(C_i - \widetilde{B}_i)\|_F \leq 32\sqrt{13\varepsilon}.$$

To finish this step, we employ the Araki-Yamagami inequality [AY81], which gives a tight modulus of continuity for the operator absolute value with respect to the Frobenius norm:

$$\||S| - |T|\|_F \leq \sqrt{2}\|S - T\|_F.$$

Since $|\lambda^* A_i| = \sqrt{A_i\lambda\lambda^* A_i} = A_i\psi_A^{1/2}A_i$ and $|C_i\lambda^*| = \sqrt{\lambda C_i^2\lambda^*} = \psi_A^{1/2}$,

$$\left\|[A_i, \psi_A^{1/2}]\right\|_F = \left\|A_i\psi_A^{1/2}A_i - \psi_A^{1/2}\right\|_F = \||\lambda^* A_i| - |C_i\lambda^*|\|_F \leq \sqrt{2}\|\lambda^* A_i - C_i\lambda^*\|_F \leq 32\sqrt{26\varepsilon}.$$

Next, we show that the A_i approximately commute. In fact, using the result of Definition 4.6,

$$\begin{aligned}\|[A_1,A_2]\psi_A^{1/2}\|_F &= \|(A_1A_2 - A_2A_1)\lambda\|_F \\ &\le \|(A_1A_2 - A_3)\lambda\|_F + \|(A_2A_1 - A_3)\lambda\|_F \\ &= \|(A_3A_1A_2 - I)\lambda\|_F + \|(A_3A_2A_1 - I)\lambda\|_F \\ &\le 376\sqrt{13\varepsilon}.\end{aligned}$$

By symmetry, this bound holds for other two commutators as well. These commutation relations also apply to the PVM elements: $\|[P^i_a, \psi_A^{1/2}]\|_F \le 16\sqrt{26\varepsilon}$ and $\|[P^i_a, P^j_b]\psi_A^{1/2}\|_F \le 94\sqrt{13\varepsilon}$.

Now, we bound the terms of $1 - w$ one by one. First, note that, for any i,

$$\langle\phi|\Pi_{\boldsymbol{a}} \otimes P^i_{\neg a_i}|\phi\rangle = \mathrm{Tr}\Big[\overline{\Pi}_{\boldsymbol{a}}\psi_A^{1/2}P^i_{\neg a_i}\psi_A^{1/2}\Big] = \Big\|P^1_{a_1}P^2_{a_2}P^3_{a_3}\psi_A^{1/2}P^i_{\neg a_i}\Big\|_F^2.$$

If $i = 3$,

$$\begin{aligned}\Big\|P^1_{a_1}P^2_{a_2}P^3_{a_3}\psi_A^{1/2}P^3_{\neg a_3}\Big\|_F &\le \Big\|P^1_{a_1}P^2_{a_2}P^3_{a_3}P^3_{\neg a_3}\psi_A^{1/2}\Big\|_F + \Big\|[P^3_{a_3}, \psi_A^{1/2}]\Big\|_F \\ &\le 16\sqrt{26\varepsilon};\end{aligned}$$

if $i = 2$,

$$\begin{aligned}\Big\|P^1_{a_1}P^2_{a_2}P^3_{a_3}\psi_A^{1/2}P^2_{\neg a_2}\Big\|_F &\le \Big\|P^1_{a_1}P^3_{a_3}P^2_{a_2}\psi_A^{1/2}P^2_{\neg a_2}\Big\|_F + \Big\|[P^2_{a_2}, P^3_{a_3}]\psi_A^{1/2}\Big\|_F \\ &\le \Big\|P^1_{a_1}P^3_{a_3}P^2_{a_2}P^2_{\neg a_2}\psi_A^{1/2}\Big\|_F + \Big\|[P^2_{a_2}, \psi_A^{1/2}]\Big\|_F + \Big\|[P^2_{a_2}, P^3_{a_3}]\psi_A^{1/2}\Big\|_F \\ &\le 16\sqrt{26\varepsilon} + 94\sqrt{13\varepsilon} = 2(8\sqrt{2} + 47)\sqrt{13\varepsilon};\end{aligned}$$

and if $i = 1$,

$$\begin{aligned}\Big\|P^1_{a_1}P^2_{a_2}P^3_{a_3}\psi_A^{1/2}P^1_{\neg a_1}\Big\|_F &\le \Big\|P^1_{a_1}P^2_{a_2}\psi_A^{1/2}P^3_{a_3}P^1_{\neg a_1}\Big\|_F + 16\sqrt{26\varepsilon} \\ &\le \Big\|P^1_{a_1}P^2_{a_2}\psi_A^{1/2}P^1_{\neg a_1}P^3_{a_3}\Big\|_F + 16\sqrt{26\varepsilon} + 94\sqrt{13\varepsilon} \\ &\le \Big\|P^2_{a_2}P^1_{a_1}\psi_A^{1/2}P^1_{\neg a_1}P^3_{a_3}\Big\|_F + 16\sqrt{26\varepsilon} + 188\sqrt{13\varepsilon} \\ &\le \Big\|P^2_{a_2}P^1_{a_1}P^1_{\neg a_1}\psi_A^{1/2}P^3_{a_3}\Big\|_F + 32\sqrt{26\varepsilon} + 188\sqrt{13\varepsilon} \\ &= 4(8\sqrt{2} + 47)\sqrt{13\varepsilon}.\end{aligned}$$

For the final term,

$$\begin{aligned}&\sum_{\substack{\boldsymbol{a}\in\{0,1\} \\ a_1+a_2+a_3=1}} \langle\phi|\Pi_{\boldsymbol{a}} \otimes I|\phi\rangle = \sum_{\substack{\boldsymbol{a}\in\{0,1\} \\ a_1+a_2+a_3=1}} \mathrm{Tr}\Big(P^3_{a_3}P^2_{a_2}P^1_{a_1}P^2_{a_2}P^3_{a_3}\psi_A\Big) \\ &= \frac{1}{32}\sum_{\substack{\boldsymbol{a}\in\{0,1\} \\ a_1+a_2+a_3=1}} \mathrm{Tr}[(I + (-1)^{a_3}A_3)(I + (-1)^{a_2}A_2)(I + (-1)^{a_1}A_1)(I + (-1)^{a_2}A_2)(I + (-1)^{a_3}A_3)\psi_A] \\ &= \frac{1}{32}\mathrm{Tr}[(4I - 4A_1A_2A_3 + 4I - 4A_2A_1A_3 + 4I - 4A_3A_1A_2 + 4I - A_3A_2A_1)\psi_A] \\ &= \frac{1}{8}\Big(\|(A_1A_2A_3 - I)\lambda\|_F^2 + \|(A_2A_1A_3 + 1)\lambda\|_F^2\Big) \\ &\le \frac{1}{4}(188\sqrt{13\varepsilon}^2) = 4 \cdot 13 \cdot 47^2\varepsilon = 114868\varepsilon.\end{aligned}$$

Putting everything together, we find the bound

$$\begin{aligned}1-w &\le \frac{4}{3}\Big((16\sqrt{26\varepsilon})^2+(2(8\sqrt{2}+47)\sqrt{13\varepsilon})^2+(4(8\sqrt{2}+47)\sqrt{13\varepsilon})^2\Big)+114868\varepsilon\\ &=\frac{1}{3}\Big(2801708+782080\sqrt{2}\Big)\varepsilon\le 1302579\varepsilon.\end{aligned}$$ ■

Now, we show the converse statement: operators satisfying a single E3-LIN relation near-perfectly induces a near-optimal strategy for the tilted bipartite cube game. This is necessary for the proof of completeness.

Lemma 4.8. Let $|\psi\rangle \in H_A \otimes H_B$ be a quantum state, and let $\{P^u_a\}_{a\in\{0,1\}} \subseteq \mathcal{B}(H_B)$ for $u \in \{011, 101, 110\}$and $\{\Pi_{\boldsymbol{a}}\}_{\boldsymbol{a}\in\{0,1\}^3} \subseteq \mathcal{B}(H_A)$ be PVMs satisfying

$$\frac{1}{3}\sum_{\substack{\boldsymbol{a}\in\{0,1\}^3\\ a_1+a_2+a_3=0}} \langle\psi|\Pi_{\boldsymbol{a}}\otimes(P^{011}_{a_1}+P^{101}_{a_2}+P^{110}_{a_3})|\psi\rangle \ge 1-\varepsilon.$$

Then, there exist PVMs $\{Q^u_a\}_{a\in\{0,1\}} \subseteq \mathcal{B}(H_A)$ for $u \in \{001, 010, 100, 111\}$ such that $s(a, b|x, y) = \langle\psi|Q^y_b \otimes P^x_a|\psi\rangle$ is a $\frac{3}{8}\varepsilon$-optimal strategy for the tilted bipartite cube game.

Proof. Without loss of generality we may suppose that $\Pi_{\boldsymbol{a}} = 0$ if $a_1 + a_2 + a_3 = 1$ — we can change Alice's PVM to one satisfying this condition without decreasing the value. Now write $\Pi^{011}_a = \sum_{a_2,a_3}\Pi_{aa_2a_3}$, $\Pi^{101}_a = \sum_{a_1,a_3}\Pi_{a_1aa_3}$, and $\Pi^{110}_a = \sum_{a_1,a_2}\Pi_{a_1a_2a}$. Let $A_u = P^u_0 - P^u_1$ and $C_u = \Pi^u_0 - \Pi^u_1$ for all $u \in \{011, 101, 110\}$; let $A_{000} = I$ and $C_{000} = C_{011}C_{101}C_{110}$; and define $B_v = \frac{1}{2}(C_{\neg v} - C_{v+e_1} - C_{v+e_2} - C_{v+e_3})$ for all $v \in X_1$. Since the C_u are commuting order-2 unitaries satisfying $C_{000}C_{011}C_{101}C_{110} = I$, the B_u are also order-2 unitaries. Next, note that

$$\begin{aligned}1-\varepsilon &= \frac{1}{3}\sum_{\substack{u\in\{011,101,110\}\\ a\in\{0,1\}}}\langle\psi|\Pi^u_a\otimes P^u_a|\psi\rangle\\ &=\frac{1}{6}\sum_{u\in X_0}(1+\langle\psi|C_u\otimes A_u|\psi\rangle)-\frac{1}{6}(1+\langle\psi|C_{000}\otimes I|\psi\rangle)\\ &=\frac{1}{2}+\frac{1}{6}\sum_{u\in X_0}\langle\psi|\frac{1}{2}(B_{\neg u}-B_{u+e_1}-B_{u+e_2}-B_{u+e_3})\otimes A_u|\psi\rangle\\ &\quad -\frac{1}{6}\sum_{\boldsymbol{a}\in\{0,1\}^3}(-1)^{a_1+a_2+a_3}\langle\psi|\Pi_{\boldsymbol{a}}\otimes I|\psi\rangle\\ &=\frac{1}{2}+\frac{4}{3}\beta(G^\Diamond,c)-\frac{1}{6}\langle\psi|I\otimes I|\psi\rangle=\frac{1}{3}+\frac{4}{3}\beta(G^\Diamond,c).\end{aligned}$$

Hence, we can rearrange and find that $\beta(G^\Diamond, c) \ge \frac{1}{2} - \frac{3}{4}\varepsilon$, as wanted. ■

Now we can pass to the proof of the main theorem.

Proof of Definition 4.1. Let (S, π) be an instance of E3-LIN. We construct a instance G of One-Sided-E1-or-2-LIN as follows. We replace each constraint with a copy of the tilted bipartite cube game, where we identify the questions 011, 101, and 110 with the original variables in the constraint, and identify the 000 variables from all the cubes, and take this to be the distinguished variable. For those constraints of the form $x_1 + x_2 + x_3 = 1$, we flip the signs of the edges adjacent to 000 in the corresponding cube.

First, we show completeness. If there exists a quantum strategy s such that $\omega(G(S,\pi), s) \geq 1-\varepsilon$. Using Naimark dilation, we may assume it is projective. Let $1 - \varepsilon_{b,\boldsymbol{x}}$ be the value on the constraint $x_1 + x_2 + x_3 = b$. Then, by constructing a strategy s' for G using Definition 4.8, the value of the corresponding cube is $\geq \frac{3}{4} - \frac{3}{8}\varepsilon_{b,\boldsymbol{x}}$, and therefore $\omega(G, s') \geq \frac{3}{4} - \frac{3}{8}\varepsilon$.

Next, we show soundness. Suppose there exists a quantum strategy s for G such that $\omega(G, s) \geq \frac{3}{4} - \delta$. As above, we can suppose the strategy is projective due to Naimark dilation. Then, via Definition 4.7, there exists a strategy s' for $G(S,\pi)$ such that $\omega(G(S,\pi), s') \geq 1 - 10^7\delta$.

Putting these together, the construction maps E3-LIN$^*_{1-\varepsilon,s}$ to One-Sided-E1-or-2-LIN$^*_{\frac{3}{4}-\frac{3}{8}\varepsilon,\frac{3}{4}-\frac{1-s}{10^7}}$. ■

5 Hardness as a function of the completeness and soundness parameters

I think that a way to deepen this theme, is to consider also the succinct setting of nonlocal games. It is true that then we will lose the clean presentation of discussing only (P vs RE) vs (P vs NP), but on the other hand we might gain a refinement that potentially will make it more relevant and appealing to the broader community of quantum/classical complexity.

Namely, in the succinct setting we not only know that XOR games are in EXP, but actually in QIP(2) (and thus also that the classical class with big gap is not only in EXP but is actually in PSPACE). To this aspect I can also contribute a result/observation I had recently, according to which AM is contained in entangled XOR games for every reasonable completeness and soundness parameters (this requires a bit more liberal definition of XOR games, see comment below definition 2.3).

In this section, we show that the best known techniques for approximating the classical value of an XOR game based on the quantum value, due to [CHTW04], extend approximation algorithms for the quantum value of tilted XOR games. We also study the range of values where Definition 4.1 implies approximating the quantum value of a tilted XOR game is hard.

Lemma 5.1. Let G be a tilted XOR game. Then, there exists a classical strategy s for G such that $\omega(G, s) \geq \frac{1}{2}$.

This also implies that E1-or-2-LIN$_{c,s}$ and E1-or-2-LIN$^*_{c,s}$ with $c \leq \frac{1}{2}$ are trivial, as every instance is a yes instance.

Proof. We may suppose without loss of generality that $\perp \notin Y$. Then, we define a classical strategy by sampling a random deterministic strategy $s(a,b|x,y) = \delta_{a,g(x)}\delta_{b,h(y)}$ for F and G uniformly random functions. Then, the bias is ≥ 0, giving that the value of the strategy is $\geq \frac{1}{2}$. ■

Lemma 5.2. Let $\gamma : [1/2, 1] \to [1/2, 1]$ be a monotone increasing function such that for all XOR games G, $\omega^*(G) \leq \gamma(\omega(G))$. Then, if $c \geq \gamma(s)$, there exist polynomial-time algorithms to decide One-Sided-E1-or-2-LIN$^*_{c,s}$, One-Sided-E1-or-2-LIN$_{c,s}$, and E1-or-2-LIN$_{c,s}$

We need only consider the γ on the domain $[1/2, 1]$ as $\omega(G) \geq \frac{1}{2}$ by the previous lemma.

Proof. Consider the following algorithm:

1. On input G compute $\omega^*(G_{XOR})$.
2. If $\omega^*(G_{XOR}) \geq c$, output YES and else output NO.

Since $\omega^*(G_{XOR})$ can be computed by an SDP, this algorithm is polynomial-time. Now, if $\omega^*(G) \geq c$, then $\omega^*(G_{XOR}) \geq \omega^*(G) \geq c$, so the algorithm outputs YES correctly. On the other hand, if $\omega^*(G) < s$, then, $\omega(G_{XOR}) = \omega(G) < s$, so using the assumption on γ, $\omega^*(G_{XOR}) < \gamma(s) \leq c$. As such, the algorithm outputs NO correctly.

The proofs for the classical cases proceed identically. ∎

The next theorem provides an explicit form of the function γ needed above.

Theorem 5.3 ([CHTW04])**.** There exists a monotone increasing function $\gamma : [1/2, 1] \to [1/2, 1]$ such that $\omega^*(G) \leq \gamma(\omega(G))$ for all XOR games G, where

$$\gamma(x) = \begin{cases} \frac{1}{2} + \kappa\left(x - \frac{1}{2}\right) & \frac{1}{2} \leq x \leq \gamma_0 \\ \gamma_1 x & \gamma_0 < x \leq \gamma_2 \\ \sin^2\left(\frac{\pi}{2}x\right) & \gamma_2 < x \leq 1, \end{cases}$$

where $\kappa \approx 1.7822$ is an upper bound on Grothendieck's constant, $\gamma_0 = \frac{\kappa-1}{2(\kappa-\gamma_1)} \approx 0.60730$ is the point where the two line segments intersect, and $\gamma_1 \approx 1.1382$ and $\gamma_2 \approx 0.74202$ are such that such that $\gamma_1 x$ is tangent to $\sin^2\left(\frac{\pi}{2}x\right)$ at $0 < \gamma_2 < 1$.

Lemma 5.4. For any $p \in (0, 1)$ and $c \geq s$, the decision problem One-Sided-E1-or-2-LIN$^*_{c,s}$ reduces to One-Sided-E1-or-2-LIN$^*_{c',s'}$ for any (c', s') that is a convex combination of (c, s), $(1, 1)$ and $(1/2, 1/2)$. The same holds for the classical problems.

Proof. The first reduction follows simply by taking a convex combination with the XOR games with winning probability 1 and $\frac{1}{2}$. ∎

The results of this section are illustrated in Figs. 1a and 1b.

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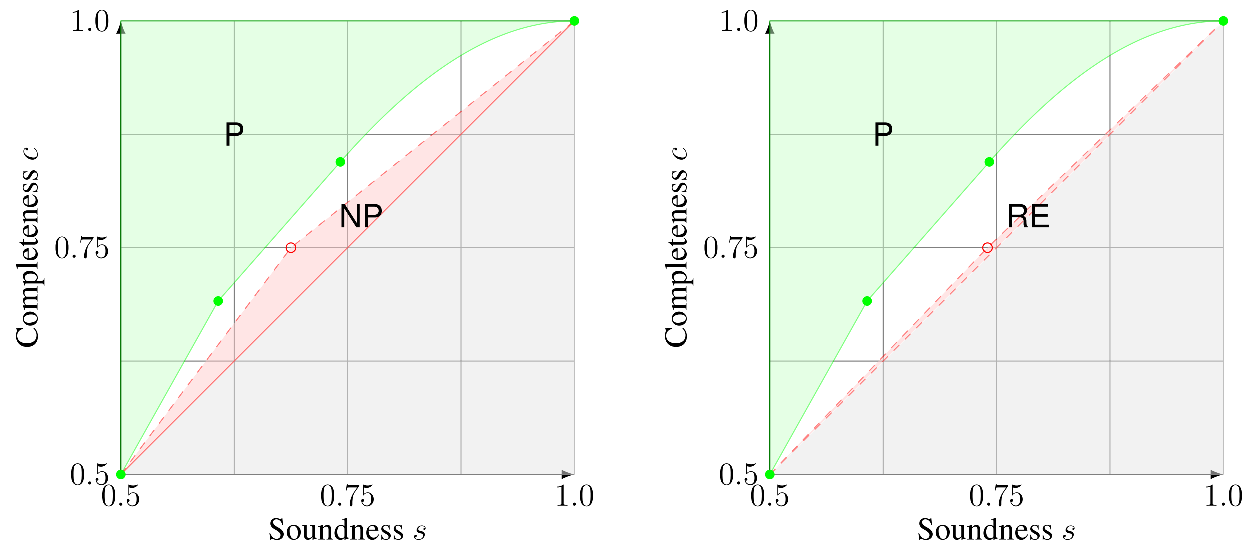

(a) Hardness of the One-Sided-E1-or-2-LIN$_{c,s}$ problems, following from Definitions 5.1 to 5.4.

(b) Hardness of the One-Sided-E1-or-2-LIN$^*_{c,s}$ problems, following from Definitions 4.1 and 5.1 to 5.4.

Figure 1: Hardness of tilted XOR games. The green region indicates problems solvable in polynomial time, the red region indicates hard instances, the white region indicates problems where the complexity is unknown, and the gray region indicates $s > c$ where the problem makes no sense. The RE-complete region is not to scale (though it is a triangle with positive area).

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