The sliding-$(ZX)^n Z$ states are graph states

J. Ding's proof as written by R. Cleve

August, 2023

The generators of the stabilizer are the form G_r for all $r \in \mathbb{Z}$, where G_r is $\cdots II(ZX)^n ZII\cdots$, positioned so that the first non-trivial Pauli is in qubit position r. For example, for $n = 2$, five of the generators of the sliding-$ZXZXZ$ state are

$$\begin{array}{rccccccccccccccc} & & {\scriptstyle -4} & {\scriptstyle -3} & {\scriptstyle -2} & {\scriptstyle -1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle 6} & {\scriptstyle 7} & {\scriptstyle 8} & \\ G_{-2} = & \cdots & I & I & Z & X & Z & X & Z & I & I & I & I & I & I & \cdots \\ G_{-1} = & \cdots & I & I & I & Z & X & Z & X & Z & I & I & I & I & I & \cdots \\ G_{0} = & \cdots & I & I & I & I & Z & X & Z & X & Z & I & I & I & I & \cdots \\ G_{1} = & \cdots & I & I & I & I & I & Z & X & Z & X & Z & I & I & I & \cdots \\ G_{2} = & \cdots & I & I & I & I & I & I & Z & X & Z & X & Z & I & I & \cdots \end{array} \tag{1}$$

For simplicity, we frequently omit leading and trailing Is when denoting Paulis.

Lemma 1. *For all even $n \geq 2$, the sliding-$(ZX)^n Z$ state is a graph state.*

Proof. The X-vectors of the above generators G_r ($r \in \mathbb{Z}$) are not full rank. We will show that, applying the Haramard H in all positions that are multiples of $m = n + 1$ apart to the generators causes the X-vectors of the generators to be of full rank, from which the result follows.

The precise local Clifford applied is

$$\begin{cases} H & \text{in all positions of the form } km - 2, \text{ for } k \in \mathbb{Z} \\ I & \text{in all other positions.} \end{cases} \tag{2}$$

Note that the effect of this local Clifford has symmetry modulo m, in the sense that its effect on the generators is the same when all the qubit positions are shifted by any multiple of m.

For convenience, we define the infinite binary basis vectors $e(i)$, for all $i \in \mathbb{Z}$, as

$$e(i)_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases} \tag{3}$$

We will consider the effect of applying the above local Clifford to the generators G_r ($r \in \mathbb{Z}$), resulting in the *modified generators* G'_r ($r \in \mathbb{Z}$). By the symmetry of the local Clifford modulo m, if $e(i)$ is in the span of the X-vectors of the modified generators then so are $e(km + i)$, for all $k \in \mathbb{Z}$.

It is useful to preform some of the analysis in terms of F_r ($r \in \mathbb{Z}$), defined as

$$\begin{aligned} F_r &= G_r\, G_{r+2} & (4) \\ &= (Z\,X\,Z\,X\,\cdots X\,Z\,X\,Z\,I\,I)(I\,I\,Z\,X\,\cdots X\,Z\,X\,Z\,X\,Z) & (5) \\ &= Z\,X\,I^{2n-1}\,X\,Z & (6) \\ &= Z\,X\,I^{2m-3}\,X\,Z \quad \text{(recall that } m = n+1), & (7) \end{aligned}$$

where the Paulis shown are in positions r to $r+2m$. Note that the X-vector of each F_r has Hamming weight two and is $e(r+1)+e(r+2m-1)$. We shall consider the effect of the local Clifford on various operators F_r, which results in $F'_r = G'_r\, G'_{r+2}$.

First, consider F'_{-1}. Since

$$F_{-1} = \overset{-1}{Z} \quad \overset{0}{X} \quad \overset{1}{I} \quad \overset{2}{I} \quad \cdots \quad I \quad I \quad \overset{2m-2}{X} \quad Z. \tag{8}$$

and the only nontrivial part of F_{-1} that is aligned with where the local Clifford applies an H is in position in $2m-2$, it holds that

$$F'_{-1} = \overset{-1}{Z} \quad \overset{0}{X} \quad \overset{1}{I} \quad \overset{2}{I} \quad \cdots \quad I \quad I \quad \overset{2m-2}{Z} \quad Z. \tag{9}$$

Therefore, the X-vector of F'_{-1} is $e(0)$. By the symmetry modulo m, for all $k \in \mathbb{Z}$, $e(km)$ is also in the span of the X-vectors of the modified generators.

Next, consider F'_1. Since

$$F_1 = \overset{1}{Z} \quad \overset{2}{X} \quad \overset{3}{I} \quad \overset{4}{I} \quad \cdots \quad I \quad I \quad \overset{2m}{X} \quad Z, \tag{10}$$

it follows that $F'_1 = F_1$ and the X-vector of F'_1 is $e(2)+e(2m)$. Since we already know that $e(2m)$ is in the span of the X-vectors of the modified generators, we can deduce that $e(2)$ is in the span of the X-vectors of the modified generators. Once again, by the symmetry modulo m, for all $k \in \mathbb{Z}$, $e(km+2)$ is also in the span of the X-vectors of the modified generators.

We can repeat this process with F'_3, F'_5, ..., F'_{m-4}, where the X-vectors are in the table

operator	X-vector
F'_{-1}	$e(0)$
F'_1	$e(2)+e(2m)$
F'_3	$e(4)+e(2m+2)$
$\vdots$	$\vdots$
F'_{m-6}	$e(m-5)+e(2m+(m-7))$
F'_{m-4}	$e(m-3)+e(2m+(m-5))$

from which it can be deduced that $e(0)$, $e(2)$, $e(4)$, ..., $e(m-3)$ (and their shifts by all multiples of m) are the the span of the X-vectors of the modified generators.

What remains is to extend this list to the remaining offsets in $\{0,1,\ldots,m-1\}$. There is one remaining even numbered offset, $m-1$ (i.e., $e(m-1)$), and there also remain all the odd numbered offsets, 1, 2, ..., $m-2$.

We begin with the sequence $e(m-4)$, $e(m-6)$, ..., $e(3)$, $e(1)$. Since

$$F_{-m-3} = Z \quad \overset{-m-2}{X} \quad I \quad \overset{-m}{I} \quad \cdots \quad I \quad I \quad \overset{m-4}{X} \quad \overset{m-3}{Z} \tag{11}$$

it follows that

$$F'_{-m-3} = Z \quad \overset{-m-2}{Z} \quad I \quad \overset{-m}{I} \quad \cdots \quad I \quad I \quad \overset{m-4}{X} \quad \overset{m-3}{Z} \tag{12}$$

whose X-vector has a single 1, in position $m-4$. Therefore, $e(m-4)$ is in the span of the X-vectors of the modified generators. And, by the symmetry modulo m, for all $k \in \mathbb{Z}$, it holds that $e(km-4)$ is in the span of the X-vectors of the modified generators.

Next, since

$$F_{-m-5} = Z \;\; \overset{-m-4}{X} \;\; I \;\; \overset{-m-2}{I} \;\; \cdots \;\; I \;\; I \;\; \overset{m-6}{X} \;\; \overset{m-5}{Z} \tag{13}$$

it follows that $F'_{-m-5} = F_{-m-5}$ and the X-vector of F'_{-m-5} is $e(m-6) + e(-m-4)$. Continuing in this manner, we obtain the results summarized in the table

operator	X-vector
F'_{-m-3}	$e(m-4)$
F'_{-m-5}	$e(m-6) + e(-m-4)$
F'_{-m-7}	$e(m-8) + e(-m-6)$
$\vdots$	$\vdots$
F'_3	$e(3) + e(-m-(m-5))$
F'_1	$e(1) + e(2m-(m-3))$

from which it can be deduced that $e(m-4)$, $e(m-6)$, $\ldots$, $e(3)$, $e(1)$ (and their shifts by all multiples of m) are the the span of the X-vectors of the modified generators.

We have now established that $e(0)$, $e(1)$, $e(2)$, $\ldots$, $e(m-4)$, $e(m-3)$ (and their shifts by all multiples of m) are the the span of the X-vectors of the modified generators. What remains is to extend this list include two more offsets: $m-2$ and $m-1$, which are handled as special cases.

First, for the offset $m-2$, consider

$$G_{-m} = \overset{-m}{Z} \;\; X \;\; Z \;\; \overset{-m+3}{X} \;\; \cdots \;\; \overset{-5}{Z} \;\; \overset{-4}{X} \;\; \overset{-3}{Z} \;\; \overset{-2}{X} \;\; \overset{-1}{Z} \;\; \overset{0}{X} \;\; \cdots \;\; \overset{m-5}{X} \;\; Z \;\; X \;\; \overset{m-2}{Z}. \tag{14}$$

Since this is aligned with the H operators in positions -2 and $m-2$, it holds that

$$G'_{-m} = \overset{-m}{Z} \;\; X \;\; Z \;\; \overset{-m+3}{X} \;\; \cdots \;\; \overset{-5}{Z} \;\; \overset{-4}{X} \;\; \overset{-3}{Z} \;\; \overset{-2}{Z} \;\; \overset{-1}{Z} \;\; \overset{0}{X} \;\; \cdots \;\; \overset{m-5}{X} \;\; Z \;\; X \;\; \overset{m-2}{X}. \tag{15}$$

Note that the X-vector of G'_{-m} has 1s in various positions in $\{0, 1, \ldots, m-3\}$ (modulo m) and in position $m-2$ (but not in position -2). Therefore, $e(m-2)$ is in the span of the X-vectors of the modified generators.

All that remains is the offset $m-1$. For this, consider

$$F_{m-2} = \overset{m-2}{Z} \;\; \overset{m-1}{X} \;\; \overset{m}{I} \;\; \overset{m+1}{I} \;\; \cdots \;\; I \;\; I \;\; X \;\; \overset{3m-2}{Z}, \tag{16}$$

which is aligned with H operators in positions $m-2$ and $3m-2$. Therefore,

$$F'_{m-2} = \overset{m-2}{X} \;\; \overset{m-1}{X} \;\; \overset{m}{I} \;\; \overset{m+1}{I} \;\; \cdots \;\; I \;\; I \;\; X \;\; \overset{3m-2}{X}, \tag{17}$$

whose X-vector has a 1 in positions $m-1$, $m-2$, $3m-3$, and $3m-2$. It follows that all of $e(0)$, $e(1)$, $e(2)$, $\ldots$, $e(m-1)$ (and their shifts by all multiples of m) are in the span of the X-vectors of the modified generators.

Therefore the X-vectors of the modified generators are full rank, which completes the proof. $\square$