Infinite stabilizer states and Clifford operations

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Abstract

These notes are in support of a talk that I will give at Perimeter on March 6 about an ongoing project with students Eric Culf, Jessie Ding, and Danny Kong.

Most of us know and love the theory of stabilizer states and Clifford operations. For example,

$$\begin{matrix} X\,X\,X \\ I\;Z\;Z \\ Z\;I\;Z \end{matrix} \qquad = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \text{(alternate notation)}$$

is a stabilizer for the 3-qubit state $|000\rangle + |111\rangle$.

I am going to define stabilizer states for a *(countably) infinite* number of qubits and consider notions of entanglement in this framework.

For these states, we start with the infinite Pauli group, whose elements are infinite tensor products of *finite weight*. For example,

$$\begin{array}{rl} X\,I\;Y\,Z\,I\;\;I\;\;I\;I\;I\;I\cdots & \text{(weight 3)} \\ -Y\,I\;\;I\;\;I\;\;I\;X\,I\;I\;I\;I\;I\cdots & \text{(weight 2)} \\ X\,X\,X\,X\,X\,X\,I\;I\;I\;I\;I\cdots & \text{(weight 6)} \end{array}$$

are such Paulis. Each pair unambiguously commutes or anticommutes. On the other hand,

$$\begin{array}{c} X\,X\,X\,X\,X\,X\,X\,X\,X\,X\cdots \\ Z\,Z\,Z\,Z\,Z\,Z\,Z\,Z\,Z\,Z\cdots \end{array}$$

are not finite weight and therefore not part of the infinite Pauli group that I am defining.

What does an infinite tensor product mean? For the time being, we can just think of these as formal objects.

I will shortly explain how *maximally commuting subgroups*[1] of the infinite Pauli group correspond to stabilizer states in a meaningful way.

[1] With the additional condition that $-I\,I\,I\cdots$ is not in the subgroup.

Big picture questions

Does this make sense as a model of states? You might wonder what the Hilbert space is.

If it does make sense, do we get a nice theory or an ungainly one? Is the theory a trivial extension of the n-qubit case, where proving the main results is very easy (and boring)? Or is it too hard to prove anything? Ideally the situation is a sweet spot between these extremes.

What is the motivation? In brief, in the finite-qubit case, there is a lot of interesting and useful theory in the restricted stabilizer/Clifford framework. Some of that theory carries over to the infinite-qubit case in an interesting way; and some of this theory is an open question. The stabilizer/Clifford framework is a potential means towards improving our understanding infinite notions of entanglement.

	finite	infinite
stabilizer states	QECCs MIP* = RE	this talk
general states	universal model	

Plan of this talk

1. Preliminary definitions.
2. Examples of such states (simple ones and exotic ones).
3. Some open questions that I hope to convince you are interesting.

Related work

Mention work on *convolutional* and *tail-biting* quantum error-correcting codes initiated by:

> M. Grassl and M. Roetteler, "Non-catastrophic encoders and encoder inverses for quantum convolutional codes." *Proc. 2006 IEEE Int. Symp. on Info. Theory*, pp. 1109–1113 (2006).
>
> G.D. Forney, M. Grassl, and S. Guha, "Convolutional and tail-biting quantum error-correcting codes." *IEEE Trans. on Info. Theory*, vol. 53, no. 3, pp. 865–880 (2007).
>
> M. Grassl and M. Roetteler, "Constructions of quantum convolutional codes." *Proc. 2007 IEEE Int. Symp. on Info. Theory (ISIT 2007)*, pp. 816–820 (2007).

These are codes for *streams* of data that encode and decode without requiring the data to be broken up into finite blocks.

The code spaces are defined in terms of infinite stabilizers with a special periodic structure. And the stabilizers are not maximal so that the associated code space is nontrivial.

1 Preliminary definitions

Returning to the infinite Pauli group, in what sense are its maximal commuting subgroups quantum states? They are abstract states on C*-algebras, which I will briefly explain.

C*-algebras

A (linear) algebra can be viewed as a set of operators acting on a vector space, or as a "free-standing" algebraic object (without having to start with a vector space).

In quantum theory, we work with Hilbert spaces instead of vector spaces. A C*-algebra is an analogous object for Hilbert spaces. It can be viewed as a set of operators acting on a Hilbert space with properties that respect the structure of the Hilbert space. Or it can be viewed as a free-standing object, with no direct reference to a Hilbert space.

C*-algebras are algebras that have a $*$-operation, and a norm with reasonable properties (that I don't list here). One key property is the C* identity: $\|a^*a\| = \|a\|^2$.

The notion of *positivity* makes sense for C*-algebras (defining $a \geq 0$ iff $a = b^*b$ for some b).

Quantum states

A key property of a quantum state is that it induces a mapping from POVM elements to probabilities. Examples: $M \mapsto \langle\psi|M|\psi\rangle$ and $M \mapsto \mathrm{Tr}(\rho M)$.

An abstract state is a "free-standing" linear functional $s : \mathcal{A} \to \mathbb{C}$ that maps POVM elements to probabilities with no reference to a Hilbert space. It should be positive (mapping positive elements of $\mathcal{A}$ to positive reals) and unital (mapping I to 1).

A POVM measurement element M also corresponds to a 2-outcome measurement with elements M and $I - M$, and it is useful to represent this as a binary observable $I - 2M$ (whose expectation is in $[-1, +1]$).

CAR-algebra

A specific C*-algebra for expressing stabilizer states is the CAR algebra (standing for "canonical anticommutation relations"). I will define the CAR algebra in a slightly different way than conventional explanations.

We can define linear combinations of elements of the Pauli group, such as

$$\begin{aligned} &\alpha(X\,I\,Y\,X\,I\,I\,I\,I\,I\,I\,I\cdots) + \beta(Y\,I\,I\,I\,I\,Z\,I\,I\,I\,I\,I\cdots) \\ &= \big(\alpha\,X\,I\,Y\,X\,I\,I + \beta\,Y\,I\,I\,I\,I\,Z\big)\,I\,I\,I\,I\,I\cdots \end{aligned}$$

Any (finite) linear combination of Paulis is of the form $M\,I\,I\,I\cdots$, for some $2^m \times 2^m$ matrix M. Define the norm of $M\,I\,I\,I\cdots$ to be $\|M\|$ (the spectral norm of M) and take the completion with respect to this norm.

The result is a C*-algebra (the CAR algebra).

Definition of state for maximal commuting stabilizer groups

For an infinite stabilizer, let $\mathcal{G}$ denote the group that it generates. We define the associated abstract state $s : \mathcal{A} \to \mathbb{C}$ first on the Pauli group $\mathcal{P}$. For all $P \in \mathcal{P}$, set

$$s(P) = \begin{cases} +1 & \text{if } P \in \mathcal{G} \\ -1 & \text{if } -P \in \mathcal{G} \\ 0 & \text{otherwise.} \end{cases}$$

Then extend the domain of s to the entire C*-algebra by linearity and continuity. This satisfies all the conditions of an abstract state, including positivity.[2]

What about evolutions in this model (e.g, unitary operations)?

So we've defined stabilizer states as abstract states, where a notion of measurement is given by the definition. What about the notion of (reversible) evolution? What we can unitary operations in the context of state vectors in a Hilbert space. The short answer is that these operations are the automorphisms of the C*-algebra (technically, called *-automorphisms). I'll say more about these operations later.

2 Examples of infinite stabilizer states

We'll start with a simple and boring warm-up example and gradually progress towards increasingly exotic states.

Example 1: every qubit in state $|0\rangle$ (informally, $|0\rangle^{\otimes\infty}$)

stabilizer	destabilizer (ignore if this is unfamiliar)
$Z\ I\ I\ I \cdots$	$X\ I\ I\ I \cdots$
$I\ Z\ I\ I \cdots$	$I\ X\ I\ I \cdots$
$I\ I\ Z\ I \cdots$	$I\ I\ X\ I \cdots$
$I\ I\ I\ Z \cdots$	$I\ I\ I\ X \cdots$
$\vdots\ \vdots\ \vdots\ \vdots\ \ddots$	$\vdots\ \vdots\ \vdots\ \vdots\ \ddots$

Minus signs flip qubits from $|0\rangle$ to $|1\rangle$. Also, swapping stabilizer/destabilizer yields $|+\rangle^{\otimes\infty}$.

[2]Danny showed positivity in his Master's thesis. Subsequently, Eric came up with a simpler proof using conditional expectation, and then I each came up with an elementary proof.

Example 2: informally $\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)^{\otimes\infty}$

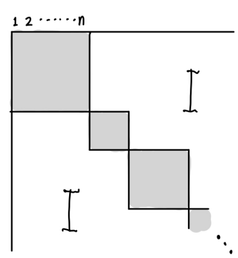
$$
\begin{array}{ccccccc}
{\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle \cdots} \\
X & X & I & I & I & I & \cdots \\
Z & Z & I & I & I & I & \cdots \\
I & I & X & X & I & I & \cdots \\
I & I & Z & Z & I & I & \cdots \\
I & I & I & I & X & X & \cdots \\
I & I & I & I & Z & Z & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
{\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle \cdots} \\
Z & I & I & I & I & I & \cdots \\
I & X & I & I & I & I & \cdots \\
I & I & Z & I & I & I & \cdots \\
I & I & I & X & I & I & \cdots \\
I & I & I & I & Z & I & \cdots \\
I & I & I & I & I & X & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Suppose that Alice possesses all the qubits in even positions and Bob possesses all the qubits in odd positions. Then they are holding a an infinite state number of Bell states. Note that such a bipartite state cannot be expressed as a vector in the tensor product of two Hilbert spaces, regardless of their dimension. So this fairly simple state is already remarkable ...

Hoe does the bipartite structure of this state work? Alice's operators are generated by those of the form of $A_0\ I\ A_2\ I\ A_4\ I \cdots$ and Bob's operators are generated by $I\ B_1\ I\ B_3\ I\ B_5\ I \cdots$.

On the other hand, this state is just an infinite concatenation of *finite* stabilizer states. One can construct many states by partitioning the infinite qubit positions into finite subsets, and then setting the qubits for each finite subset to an ordinary stabilizer state.

Thus, the stabilizer would consist of infinitely many square blocks, each of finite size:

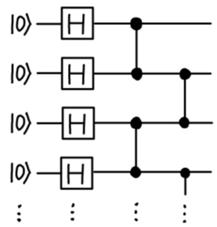


Can we come up with a state that is not of this form?

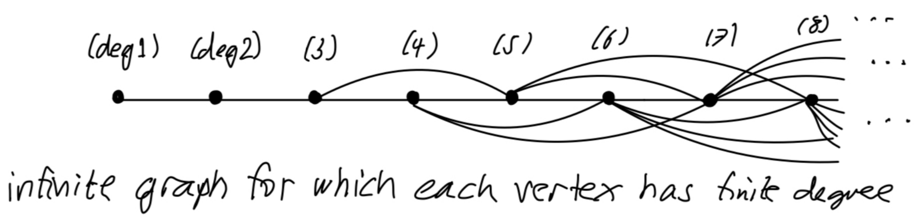
Example 3: the infinite line-graph state

$$
\begin{array}{cccccccc}
X & Z & I & I & I & I & I & \cdots \\
Z & X & Z & I & I & I & I & \cdots \\
I & Z & X & Z & I & I & I & \cdots \\
I & I & Z & X & Z & I & I & \cdots \\
I & I & I & Z & X & Z & I & \cdots \\
I & I & I & I & Z & X & Z & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
Z & I & I & I & I & I & \cdots \\
I & Z & I & I & I & I & \cdots \\
I & I & Z & I & I & I & \cdots \\
I & I & I & Z & I & I & \cdots \\
I & I & I & I & Z & I & \cdots \\
I & I & I & I & I & Z & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

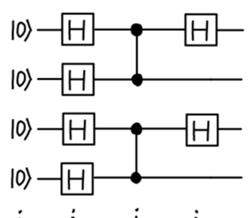
This is not a collection of independent finite-qubit states. It's the graph state of the infinite line: start with $|+\rangle^{\otimes\infty}$ and apply a controlled-Z gate between qubit k and $k+1$ for all k. The process can be illustrated by the following infinite circuit diagram:



For *any* graph on infinitely many vertices of finite-degree, there is a stabilizer state corresponding to that graph state. For example, the graph could be the following one, where vertex k has degree k (for all k):



And the state of Example 2 is also a graph state (not a strict graph state, but a graph state with an additional layer of 1-qubit Clifford operations):



All these states are still kind of tame ...

Consider the next example.

Example 4: sliding-XYZ state

$$\begin{matrix} X & Y & Z & I & I & I & I & I & \cdots \\ I & X & Y & Z & I & I & I & I & \cdots \\ Z & I & X & Y & Z & I & I & I & \cdots \\ Z & Z & I & X & Y & Z & I & I & \cdots \\ Z & Z & Z & I & X & Y & Z & I & \cdots \\ Z & Z & Z & Z & I & X & Y & Z & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

By staring at it for a little while, you can convince yourself that these generators are commuting. It is not immediately obvious that this generates a *maximal* commuting subgroup—but it turns out that it is maximal.

The destabilizer is not shown. Can you see what it is?

Recall that, for finitely many qubits, all stabilizer states are graph states (up to single qubit Clifford operations). Question: it this state graph state (up to 1-qubit unitaries)?

We'll get back to these questions ...

3 Open questions

Open question 1: is every state a graph state (up to qubit-local Cliffords)?

Clearly the first four examples are graph states.

How does the proof work for n qubits? The standard proof goes like this:

$$\begin{matrix} \boldsymbol{X} & X & X \\ I & Z & Z \\ Z & I & Z \end{matrix} \mapsto \begin{matrix} \boldsymbol{X} & Z & X \\ I & \boldsymbol{X} & Z \\ Z & I & Z \end{matrix} \mapsto \begin{matrix} \boldsymbol{X} & Z & Z \\ I & \boldsymbol{X} & X \\ Z & I & \boldsymbol{X} \end{matrix}$$

The last stabilizer can be expressed in matrix form as

$$[X\,|Z] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right], \quad \text{which is equivalent to} \quad [X\,|Z] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right].$$

Once the X-matrix in upper triangular form with 1s on the diagonal, it has full rank. Therefore, there exist generators for which that X-matrix is the identity.

The commutativity relations imply that Z-matrix is symmetric (the adjacency matrix of a graph), so it's a graph state.

Let's try this proof proof for the sliding-XYZ state ...

The X-matrix is already in upper triangular form with Xs on the diagonal:

$$[X\,|Z] = \left[\begin{array}{c|c} 1\,1\,0\,0\,0\,0\,0\cdots & 0\,1\,1\,0\,0\,0\,0\cdots \\ 0\,1\,1\,0\,0\,0\,0\cdots & 0\,0\,1\,1\,0\,0\,0\cdots \\ 0\,0\,1\,1\,0\,0\,0\cdots & 1\,0\,0\,1\,1\,0\,0\cdots \\ 0\,0\,0\,1\,1\,0\,0\cdots & 1\,1\,0\,0\,1\,1\,0\cdots \\ 0\,0\,0\,0\,1\,1\,0\cdots & 1\,1\,1\,0\,0\,1\,1\cdots \\ \vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\ddots & \vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\ddots \end{array}\right]$$

Nevertheless, the X-matrix does not have full rank! Every vector in the span of the rows of the X-matrix has an even number of 1s. So the standard proof, that works for finitely many qubits, breaks down.

Neither the X-matrix, the Z-matrix, nor the Y-matrix have full rank. So maybe this is not a graph state? What do you think?

It turns out that it is a graph state.

Define $C = HS^*$ (this cyclically permutes the X, Y, and Z Paulis). Then apply these Cliffords to the qubit positions:

$$C\ I\ I\ I\ H\ C\ I\ H\ C\ I\ H\ C\ I\cdots\ = C\ I\ I\ (\ I\ H\ C\ )^{\otimes\infty}.$$

The resulting X-matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which turns out to have full rank (though this is not obvious from a glance). How were these local Cliffords were found? In an ad hoc manner. I was trying to prove that no local Cliffords would result in a full rank X-matrix and ended up with the above Cliffords as a counterexample. Therefore, the sliding-XYZ state is a graph state.

After this example, we (with Eric and Jessie) have investigated several other states that, at first, looked like they might not be graph states. However, upon investigation, they all turned out to be graph states.

But we don't know of a *systematic* conversion method to graph states. We do not know whether or not all infinite stabilizer states are graph states.

Incidentally, the stabilizer for example 4 has no destabilizer. However, Eric showed that, for all stabilizer states, there always exists *some* set of generators of the stabilizer group that has a destabilizer. Note the contrast with the finite case, where *every* stabilizer has a destabilizer.

Open question 2: what is the Clifford group for infinitely many qubits?

For any C*-algebra $\mathcal{A}$, we can define $u \in \mathcal{A}$ to be *unitary* if $uu^* = u^*u = I$.

How does a unitary operation act on states? In the Heisenberg picture, u acts on measurement elements by conjugation: $M \mapsto u^*Mu$. This induces an automorphism (technically, called a *-automorphism) on $\mathcal{A}$. These are called *inner automorphisms.*

In general, C*-algebras have automorphisms that are not expressible this way, which are called *outer isomorphisms.*

These automorphisms are the *evolutions* of abstract states on C*-algebras. They are all the reversible operations that respect the structure of the C*-algebra.

The *Clifford evolutions* are the specific automorphisms that act as permutations on the Pauli group (such automorphisms preserve commutations and anticommutations).

Qubit-local Clifford operations

One example, is $H\ H\ H\ H\ H \cdots$, which induces a simple-to-describe permutation on $\mathcal{P}$.

In fact, *any* infinite tensor product of 1-qubit Clifford operations (the group generated H and S) induces a valid Clifford operation. Note that some of them are inner automorphisms and some are outer automorphisms.

Graph-controlled-Z Clifford operations

Another type of Clifford evolution, which implicitly arose in the context of graph states, is as follows. Take any group with infinitely many vertices and such that, for each vertex, it's degree is finite.

Consider a quantum circuit with infinitely many wires, and insert a controlled-Z gate between wire i and wire j if and only if (i, j) is an edge of the graph. Since the controlled-Z gates all commute, the order of the gates does not matter. It's not too hard to see that this induces a permutation on the Pauli group, and is a Clifford operation.

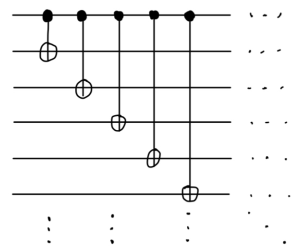
Now, imagine composing the above two types of Clifford operations (qubit local and graph-controlled$Z$) any finite number of times. The result is a group that is certainly a *subgroup* of the infinite Clifford group.

Open question: is this the entire Clifford group, or are there other, more exotic, Clifford operations?

Is there a Clifford operation that cannot be expressed as a finite alternation of qubit-local Cliffords and graph-controlled-Z operations?

Given that we can visualize Graph-controlled-Z operations as "infinite circuits", we might consider whether other infinite circuit diagrams make sense as Clifford operations (or, more generally, as automorphisms of the CAR algebra).

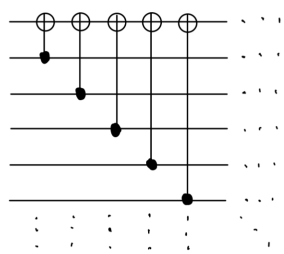
Consider this infinte circuit:



It makes perfect sense as an invertible mapping on the "computational basis[3] states"

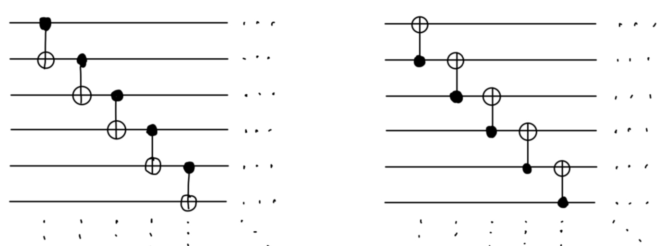
$$|b_0\rangle|b_1\rangle|b_2\rangle|b_3\rangle\cdots \mapsto |b_0\rangle|b_1\oplus b_0\rangle|b_2\oplus b_0\rangle|b_3\oplus b_0\rangle\cdots, \tag{1}$$

however, it makes no sense in the Hadamard basis where it becomes this circuit.



Think of what $|0\rangle|1\rangle|1\rangle|1\rangle\cdots$ could map to. And it does not satisfy the definition of not a permutation on the Pauli group.

What about these two circuits?



They make each sense as invertible operations in the computational basis as well as the Hadamard basis. Nevertheless, they are not valid automorphisms of the Pauli group. The first one maps $X\ I\ I\ I\ I\cdots$ to $X\,X\,X\,X\,X\cdots$, which is not in the Pauli group.

The point is that it's not so easy to construct a valid Clifford operation that does not consist of a finite number of alternations of qubit-local Cliffords and graph-controlled-Z operations.

[3]Which are actually not a basis for the set ofinfinite stabilizer states.

Open question 3: are all bipartite states tensor products of Bell states?

Depending on the answers to the first two open questions, the infinite of Stabilizer/Clifford world might be:

tame: with all stabilizer states equivalent to graph states and all Clifford operations consisting of finitely many alternations of two simple types of Clifford operations.

wild: with more exotic types of stabilizer states and Clifford operations arising.

Even in the tame case, there is this open question: are all bipartite stabilizer states equivalent to a tensor product of (finitely many or infinitely many) Bell states?

The obvious starting point for addressing this is to look at the proof that, for finitely many qubits, all bipartite states are tensor products of Bell states (up to local Clifford operations on each side of the bipartition). That proof starts by putting the stabilizer into the form of a graph state.

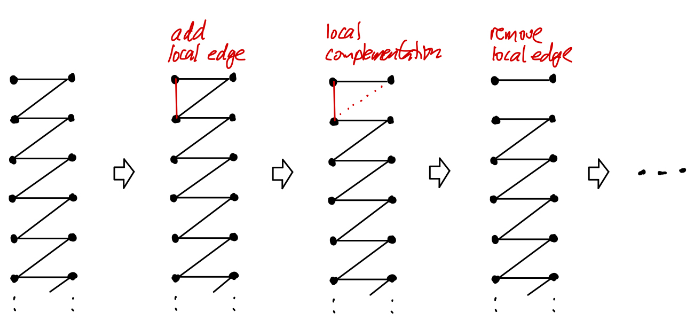
There are certain operations that can be locally performed on the qubits that modify the edges of the graph:

adding/removing edges within one side of the bipartition: Alice can add or remove an edge from this graph if both ends of the edge are in her part of the bipartition (and similarly for Bob).

local complementation: enables certain transformations of a graph to be performed by applying local Cliffords to qubits.

It turns out that one can start with an arbitrary graph state on n vertices, and perform a series of operations of the above two types so as to remove all edges, except for a matching between vertices on Alice's side and Bob's side. That graph state essentially corresponds to a series of Bell states between Alice and Bob.

What happens if we perform these operations for one of the infinite graph states described before? One can come up with an *infinite sequence* of operations of the above type reduce an arbitrary graph into a matching.



But do the infinite sequences of operations make sense as Clifford operations? It turns out that the infinite sequence corresponding to the infinite process resembles the infinite circuit diagrams at the bottom of page 10 (with the "infinite staircase" of CNOT gates), which is not in the Clifford group (nor does it induce an automorphism on the CAR-algebra). So the standard approach that works for finitely many qubits does not work.

However, it does not rule out the possibility that there is some other approach where Alice and Bob's operations are local Clifford operations.

4 Further questions

Everything that I have spoken about so far concerns *maximal* stabilizers—which correspond to pure states.[4] There are also results and open problems outside this setting.

Non-maximal stabilizers

Non-maximal stabilizers ought to correspond to code spaces. However, there are some technicalities that arise in defining such code spaces in the abstract state formalism. The code spaces make sense if there are additional assumptions made about the stabilizer; however, in general, it is not clear how to define code space for arbitrary non-maximal stabilizers. I have made some partial progress on the matter.

Mixed stabilizer states

It also makes sense to define a *mixed* stabilizer state. For finitely many qubits, one can think of such a state as the result of a pure stabilizer state on a larger set of qubits, with the additional qubits traced out. In our infinite model, the notion of a partial trace is straightforward to define (where the number of qubits traced out can be a finite or infinite subset of all the qubit positions). Therefore, such a definition of mixed stabilizer state makes sense. There are some subtleties that arise with the notion of purification, because some natural probabilistic mixtures of pure states cannot be expressed as pure states on a system with additional qubits. One way of accommodating these states is by extending the CAR algebra via a construction known as the *-crossed product.

[4]Proven to me by Eric and independently by Jessica.