Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lecture 9 (2019)

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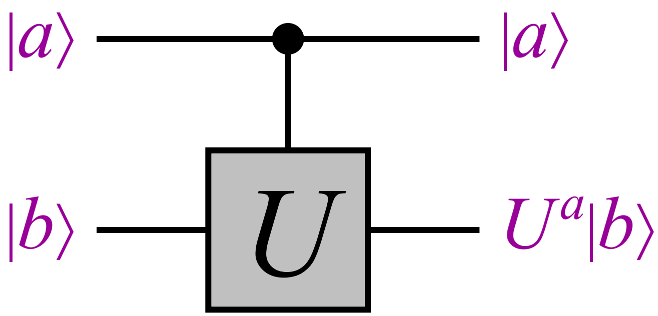
Recap of:

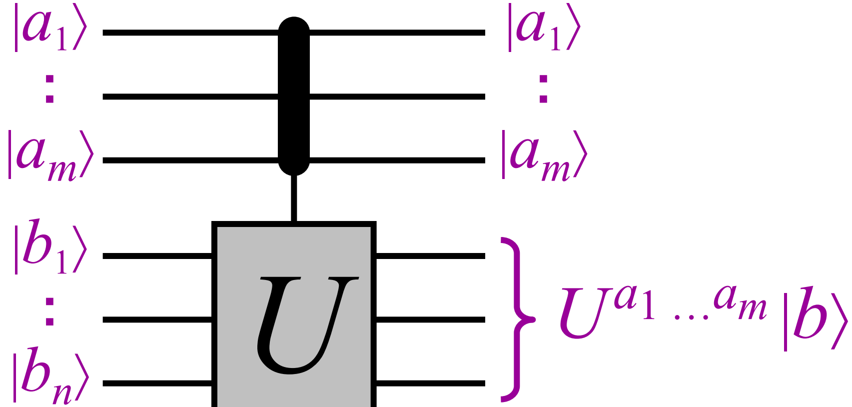
Eigenvalue estimation problem

(a.k.a. phase estimation)

Generalized controlled-U gates



$$\begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}$$



$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & U & 0 & \cdots & 0 \\ 0 & 0 & U^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & U^{2^m-1} \end{bmatrix}$$

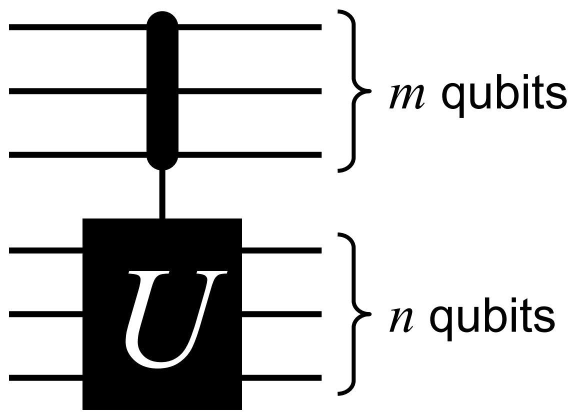
Example: $|1101\rangle|0101\rangle \rightarrow |1101\rangle U^{1101}|0101\rangle$

Eigenvalue estimation problem

U is a unitary operation on n qubits

$|\psi\rangle$ is an eigenvector of U, with eigenvalue $e^{2\pi i\phi}$ $(0 \leq \phi < 1)$

Input: black-box for



and a copy of $|\psi\rangle$

Output: ϕ (m-bit approximation)

Algorithm:
- one query to generalized controlled-U gate
- $O(n^2)$ auxiliary gates
- Success probability $4/\pi^2 \approx 0.4$

Note: with **$2m$-qubit** control gate, error probability is exponentially small

Order-finding via eigenvalue estimation

Order-finding problem

Let m be an n-bit integer (***not*** necessarily prime)

Def: $\mathbb{Z}_m^{\;*} = \{x \in \{1,2,\ldots, m-1\} : \gcd(x,m) = 1\}$ a group (mult.)

Def: $\mathrm{ord}_m(a)$ is the minimum $r > 0$ such that $a^r = 1 \pmod{m}$

Order-finding problem: given m and $a \in \mathbb{Z}_m^{\;*}$ find $\mathrm{ord}_m(a)$

Example: $\mathbb{Z}_{21}^{\;*} = \{1,2,4,5,8,10,11,13,16,17,19,20\}$

The powers of 5 are: $1, 5, 4, 20, 16, 17, 1, 5, 4, 20, 16, 17, 1, 5, \ldots$

Therefore, $\mathrm{ord}_{21}(5) = 6$

Note: no ***classical*** polynomial-time algorithm is known for this problem—it turns out that this is as hard as factoring

Order-finding algorithm (1)

Define: U (an operation on n qubits) as: $U|y\rangle = |a\,y \bmod m\rangle$

Define: $$|\psi_1\rangle = \sum_{j=0}^{r-1} e^{-2\pi i(1/r)j} \left|a^j \bmod m\right\rangle$$

Then $$U|\psi_1\rangle = \sum_{j=0}^{r-1} e^{-2\pi i(1/r)j} \left|a^{j+1} \bmod m\right\rangle$$

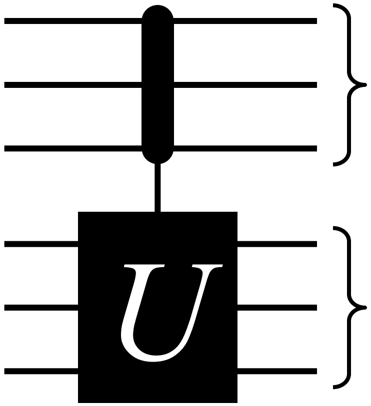
$$= \sum_{j=0}^{r-1} e^{2\pi i(1/r)} e^{-2\pi i(1/r)(j+1)} \left|a^{j+1} \bmod m\right\rangle$$

$$= e^{2\pi i(1/r)} |\psi_1\rangle$$

Therefore $|\psi_1\rangle$ is an eigenvector of U

And knowing the eigenvalue is equivalent to knowing $1/r$, from which r can be determined

Order-finding algorithm (2)



corresponds to the mapping:

$|x\rangle|y\rangle \mapsto |x\rangle|a^x y \bmod m\rangle$

Moreover, this mapping can be implemented with $O(n^2 \log n)$ gates (n multiplications in "repeated squaring" algorithm)

The eigenvalue estimation algorithm yields a $2n$-bit estimate of $1/r$ (using the above mapping and the state $|\psi_1\rangle$)

From this, a good estimate of r can be calculated by taking the reciprocal, and rounding off to the nearest integer

Exercise: why are $2n$ bits necessary and sufficient for this?

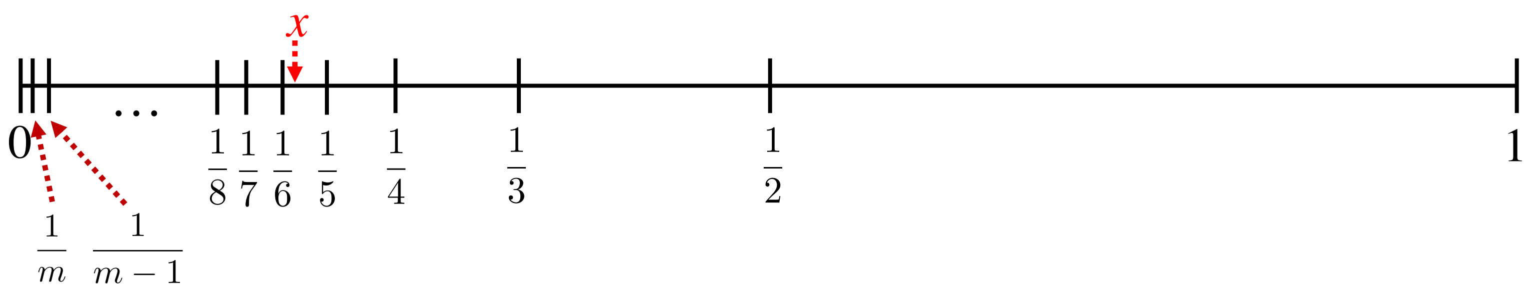
Big problem: how do we construct state $|\psi_1\rangle$ to begin with?

* we're now using m for the modulus and setting the number of control qubits to $2n$

Order-finding algorithm (3)

Solution to exercise: If $r \in \{1, 2, \ldots, m\}$, where m is an n-bit integer then a $2n$-bit approximation of $1/r$ is necessary and sufficient to determine r

The obvious procedure is to check where x lands on the line and round to the nearest $1/r$



The hardest case to distinguish is between $1/m$ vs $1/(m-1)$, where the gap is

$$\frac{1}{m-1} - \frac{1}{m} = \frac{1}{(m-1)m}$$

When $m = 2^n$, this gap is $\frac{1}{(2^n - 1)2^n} \approx \frac{1}{2^{2n}}$

This is the basic idea why $2n$ bits precision is necessary and sufficient

Bypassing the need for $|\psi_1\rangle$ (1)

Note: if we let

$$|\psi_1\rangle = \sum_{j=0}^{r-1} e^{-2\pi i(1/r)j} |a^j \bmod m\rangle$$

$$|\psi_2\rangle = \sum_{j=0}^{r-1} e^{-2\pi i(2/r)j} |a^j \bmod m\rangle$$

$$\vdots$$

$$|\psi_k\rangle = \sum_{j=0}^{r-1} e^{-2\pi i(k/r)j} |a^j \bmod m\rangle$$

$$\vdots$$

$$|\psi_r\rangle = \sum_{j=0}^{r-1} e^{-2\pi i(r/r)j} |a^j \bmod m\rangle$$

then ***any*** one of these could be used in the previous procedure, yielding an estimate of k/r, from which r can be extracted

What if k is chosen randomly and kept secret?

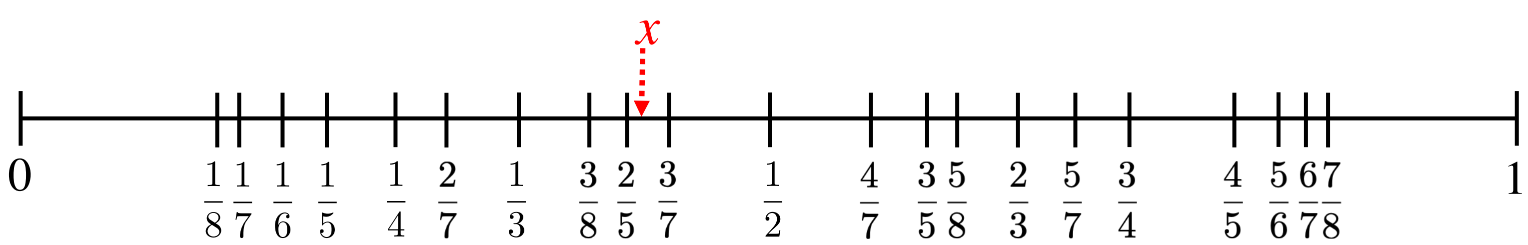
Bypassing the need for $|\psi_1\rangle$ (2)

What if k is chosen randomly and kept secret?

Let $r \in \{1, 2, \ldots, m\}$, where m is an n-bit integer and $k \in \{1, 2, \ldots, r\}$

Given x, a $2n$-bit approximation of k/r, can we determine k, r ?

The situation is now more complicated, though in principle we could still imagine checking where x lands on the line and round to the nearest k/r



The hardest case to distinguish is still between $1/m$ vs $1/(m-1)$, where the gap is around $1/m^2$

Of course, we cannot distinguish between these r/k: $1/2 = 2/4 = 3/6 = 4/8$, but at least the procedure makes sense when $\gcd(k, r) = 1$

But: is there an algorithm that finds k and r in time polynomial in n?

Bypassing the need for $|\psi_1\rangle$ (3)

Let $r \in \{1, 2, \ldots, m\}$, where m is an n-bits, $k \in \{1, 2, \ldots, r\}$, where $\gcd(k,r)=1$

Question: given x, a $2n$-bit approximation of k/r, is there an efficient algorithm to determine k, r? (i.e. by *efficient*, we mean time polynomial in n)

Answer: Yes, the ***continued fractions algorithm**** does exactly this!

* For a discussion of the *continued fractions algorithm*, please see Appendix A4.4 in [Nielsen & Chuang]

Bypassing the need for $|\psi_1\rangle$ (4)

What is the probability that k and r are relatively prime?

Recall that k is randomly chosen from $\{1,\ldots,r\}$

The probability that this occurs is $\phi(r)/r$, where ϕ is ***Euler's totient function*** (which is defined as the size of $\mathbb{Z}_r^*$)

It is known that $\phi(r) = \Omega(r/\log\log r)$, which implies that the above probability is at least $\Omega(1/\log\log r) = \Omega(1/\log n)$

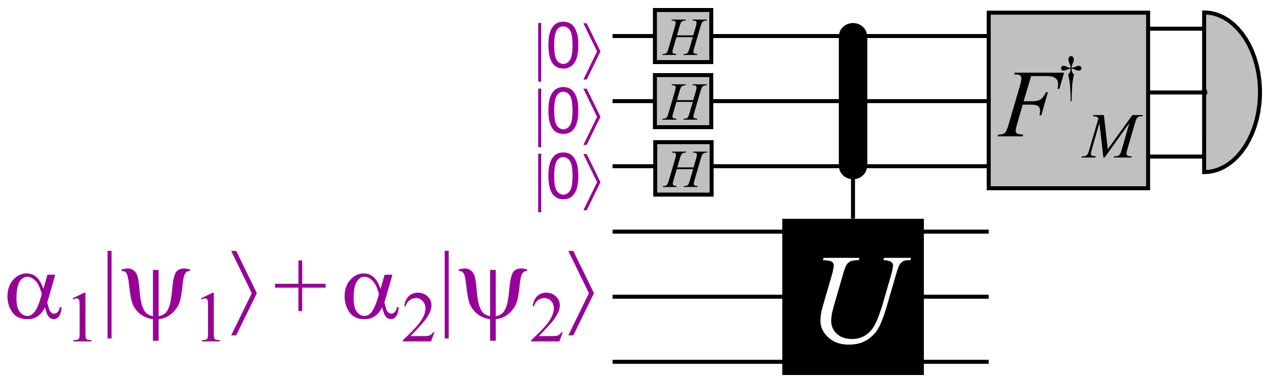
Therefore, the success probability is at least $\Omega(1/\log n)$

Is this good enough? Yes, because it means that the success probability can be amplified to any constant < 1 by repeating $O(\log n)$ times (so still polynomial in n)

But we'd still need to generate a random $|\psi_k\rangle$ here …

Bypassing the need for $|\psi_1\rangle$ (5)

Returning to the phase estimation problem, suppose that $|\psi_1\rangle$ and $|\psi_2\rangle$ have respective eigenvalues $e^{2\pi i\phi_1}$ and $e^{2\pi i\phi_2}$, and that $\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle$ is used in place of an eigenvector:



What will the outcome of the measurement be?

It can be shown* that the outcome will be an estimate of

$$\begin{cases} \phi_1 \text{ with probability } |\alpha_1|^2 \\ \phi_2 \text{ with probability } |\alpha_2|^2 \end{cases}$$

* Showing this is straightforward, but not entirely trivial

Bypassing the need for $|\psi_1\rangle$ (6)

Along these lines, using $\frac{1}{\sqrt{r}}\sum_{k=1}^{r}|\psi_k\rangle$ yields the same outcome as using a random $|\psi_k\rangle$ (but not being given k), where each $k \in \{1,\ldots,r\}$ occurs with probability $1/r$ — this is a case that we've already solved

So now all we have to do is construct the state $\frac{1}{\sqrt{r}}\sum_{k=1}^{r}|\psi_k\rangle$

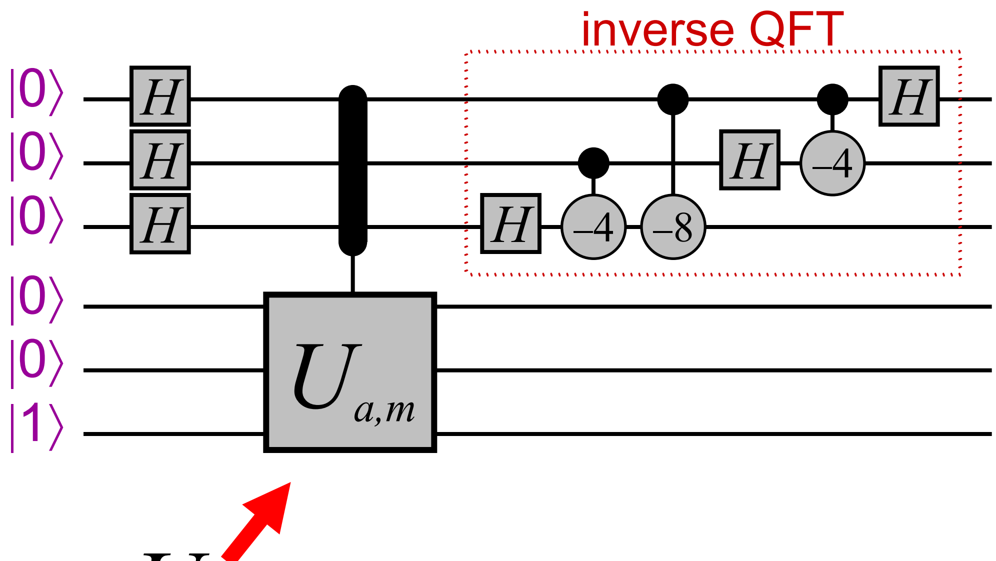
Since

$$
\begin{aligned}
\frac{1}{\sqrt{r}}\sum_{k=1}^{r}|\psi_k\rangle = & \frac{1}{\sqrt{r}}\frac{1}{\sqrt{r}}\Big(|1\rangle + \omega^{-1}|a\rangle + \omega^{-2}|a^2\rangle + \cdots + \omega^{-(r-1)}|a^{r-1}\rangle\Big) \\
& + \frac{1}{\sqrt{r}}\frac{1}{\sqrt{r}}\Big(|1\rangle + \omega^{-2}|a\rangle + \omega^{-4}|a^2\rangle + \cdots + \omega^{-2(r-1)}|a^{r-1}\rangle\Big) \\
& + \frac{1}{\sqrt{r}}\frac{1}{\sqrt{r}}\Big(|1\rangle + \omega^{-3}|a\rangle + \omega^{-6}|a^2\rangle + \cdots + \omega^{-3(r-1)}|a^{r-1}\rangle\Big) \\
& + \quad \vdots \quad\; \vdots \qquad\qquad\qquad\qquad \vdots \\
& + \frac{1}{\sqrt{r}}\frac{1}{\sqrt{r}}\Big(|1\rangle + \omega^{-r}|a\rangle + \omega^{-2r}|a^2\rangle + \cdots + \omega^{-r(r-1)}|a^{r-1}\rangle\Big) = |1\rangle
\end{aligned}
$$

its easy!

This is how the previous requirement for $|\psi_1\rangle$ is bypassed

Quantum algorithm for order-finding



measure these qubits and apply ***continued fractions algorithm*** to determine (with constant success probability) k' and r' such that $k'/r' = k/r$

$$U_{a,m}|y\rangle = |ay \bmod m\rangle$$

Number of gates for $\Omega(1/\log n)$ success probability is: $O(n^2 \log n)$
(this is the cost of $O(n)$ multiplications)

For any ***constant*** success probability, repeat $O(\log n)$ times and take the smallest resulting r' that satisfies the equation $a^{r'} = 1 \pmod m$

Reduction from factoring to order-finding

The integer factorization problem

Input: m (n-bit integer; we can assume it is composite)

Output: p, q (each greater than 1) such that $pq = m$

Note 1: no efficient (polynomial-time) classical algorithm is known for this problem

Note 2: given any efficient algorithm for the above, we can recursively apply it to fully factor m into primes* efficiently

* A polynomial-time ***classical*** algorithm for ***primality testing*** exists

Factoring prime-powers

There is a straightforward ***classical*** algorithm for factoring numbers of the form $m = p^k$, for some prime p

What is this algorithm?

Therefore, the interesting remaining case is where m has at least two distinct prime factors

Numbers other than prime-powers

Proposed quantum algorithm (repeatedly do):

1. randomly choose $a \in \{2, 3, \ldots, m-1\}$
2. compute $g = \gcd(a, m)$
3. **if** $g > 1$ **then**

   output g, m/g

   else

   compute $r = \text{ord}_m(a)$ (quantum part)

   if r is even **then**

   compute $x = a^{r/2} - 1 \bmod m$

   compute $h = \gcd(x, m)$

   if $h > 1$ **then** output h, m/h

Analysis:

we have $m \mid a^r - 1$

so $m \mid (a^{r/2}+1)(a^{r/2}-1)$

thus, either $m \mid a^{r/2}+1$
or $\gcd(a^{r/2}+1, m)$
is a nontrivial factor of m

It can be shown that at least half of the $a \in \{2, 3, \ldots, m-1\}$ have even order and result in $\gcd(a^{r/2}+1, m)$ being a nontrivial factor of m