Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lectures 6–8 (2019)

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Discrete log problem

Discrete logarithm problem (DLP)

Input: p (prime), g (generator of $\mathbb{Z}_p$*), $a \in \mathbb{Z}_p$*

Output: $r \in \mathbb{Z}_{p-1}$ such that $g^r \bmod p = a$

Example: $p = 7$, $\mathbb{Z}_7$* $= \{1, 2, 3, 4, 5, 6\} = \{3^0, 3^2, 3^1, 3^4, 3^5, 3^3\}$ (hence 3 is a generator of $\mathbb{Z}_7$*)

For $a = 6$, since $3^3 = 6$, the output should be $r = 3$

Note: No efficient classical algorithm for ***DLP*** is known (and cryptosystems exist whose security is predicated on the computational difficulty of DLP)

Efficient quantum algorithm for DLP?
(**Hint:** it can be made to look like Simon's problem!)

DLP similar to Simon's problem

Clever idea (of Shor): define $f: \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1} \rightarrow \mathbb{Z}_p^*$ as

$f(x_1, x_2) = g^{x_1} a^{-x_2} \bmod p$ (can be efficiently computed)

When is $f(x_1, x_2) = f(y_1, y_2)$?

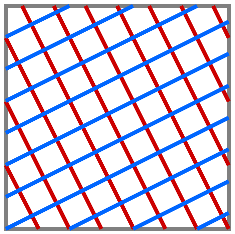
We know $a = g^r$ for ***some*** r, so $f(x_1, x_2) = g^{x_1 - r x_2} \bmod p$

Thus, $f(x_1, x_2) = f(y_1, y_2)$ iff $x_1 - r x_2 \equiv y_1 - r y_2 \pmod{p-1}$

iff $(x_1, x_2)\cdot(1, -r) \equiv (y_1, y_2)\cdot(1, -r) \pmod{p-1}$

iff $((x_1, x_2) - (y_1, y_2))\cdot(1, -r) \equiv 0 \pmod{p-1}$

iff $(x_1, x_2) - (y_1, y_2) \equiv k(r, 1) \pmod{p-1}$



Recall Simon's property: $f(x) = f(y)$ iff $x - y = kr \pmod{2}$

Simon's problem modulo m

The function arising in DLP can be abstracted to the following

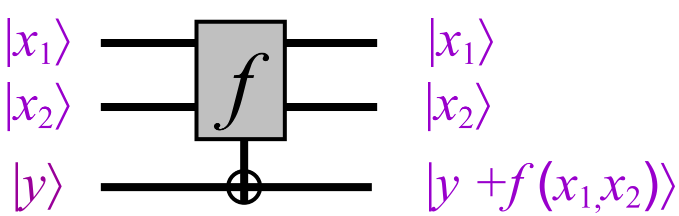
Given: $f: \mathbb{Z}_m \times \mathbb{Z}_m \rightarrow T$ with the property that:

$f(x_1, x_2) = f(y_1, y_2)$ iff $(x_1, x_2) - (y_1, y_2) \equiv k(r_1, r_2) \pmod{m}$
where (r_1, r_2) is the hidden data

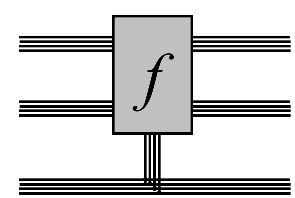
Goal: determine (r_1, r_2)

Note: in DLP case, $(r_1, r_2) = (r, 1)$

The reversible query box for f is:



where each "wire" denotes many qubit wires, to represent elements of $\mathbb{Z}_m$ like:



Not a "black" box, because we can simulate it by 1-qubit and 2-qubit gates (and this can be done efficiently) …

Digression:
on simulating black boxes

How *not* to simulate a black box

Given an efficiently (classically) computable function, over some finite domain, such as $f(x) = g^{x_1} a^{-x_2} \bmod p$, simulate f-queries over that domain

Easy to compute mapping $|x\rangle|y\rangle|00...0\rangle \mapsto |x\rangle|y \oplus f(x)\rangle|g(x)\rangle$, where the third register is "work space" with accumulated "garbage" (e.g., two such bits arise when a Toffoli gate is used to simulate an AND gate)

This works fine – ***as long as f is not queried in superposition***

If f is queried in superposition then the resulting state can be $\sum_x \alpha_x |x\rangle|y \oplus f(x)\rangle|g(x)\rangle$ can we just discard the third register?

No ... there could be entanglement ...

How *to* simulate a black box

Simulate the mapping $|x\rangle|y\rangle|00...0\rangle \mapsto |x\rangle|y\oplus f(x)\rangle|00...0\rangle$, (i.e., clean up the "garbage")

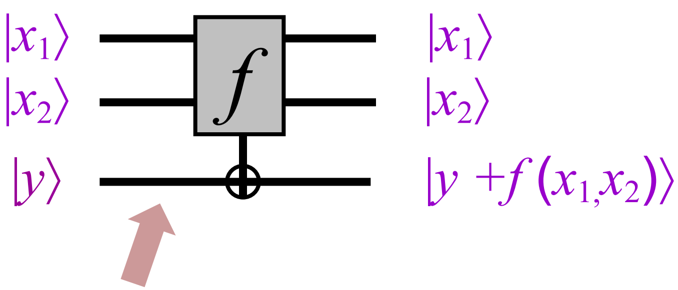
To do this, use an additional register, and:

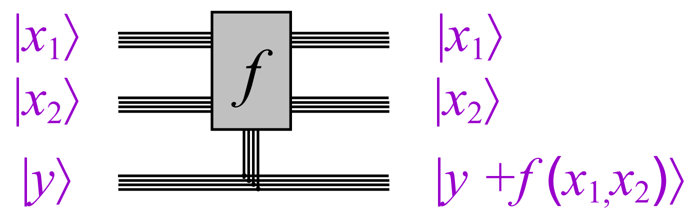
1. compute $|x\rangle|y\rangle|00...0\rangle|00...0\rangle \mapsto |x\rangle|y\rangle|f(x)\rangle|g(x)\rangle$ (ignoring the 2nd register in this step)
2. compute $|x\rangle|y\rangle|f(x)\rangle|g(x)\rangle \mapsto |x\rangle|y\oplus f(x)\rangle|f(x)\rangle|g(x)\rangle$ (using CNOT gates between the 2nd and 3rd registers)
3. compute $|x\rangle|y\oplus f(x)\rangle|f(x)\rangle|g(x)\rangle \mapsto |x\rangle|y\oplus f(x)\rangle|00...0\rangle|00...0\rangle$ (by reversing the procedure in step 1)

Total cost: around twice the classical cost of computing f, plus n auxiliary CNOT gates

Simon's problem modulo *m*

So now we have an efficient way of implementing the reversible black box for f





Reminder: each "thick wire" denotes several qubits, to represent an element of $\mathbb{Z}_m$ (eg, $\{0, 1, 2, 3, 4, 5, 6\} = \{000, 001, 010, 011, 100, 101, 110\}$)

OK, so what about a quantum algorithm for this problem?

To get one, we go beyond the Hadamard transform, which has been our main tool so far, to …

Quantum Fourier transform (QFT)

Quantum Fourier transform

$$F_m = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{m-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(m-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(m-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{m-1} & \omega^{2(m-1)} & \omega^{3(m-1)} & \cdots & \omega^{(m-1)^2} \end{bmatrix}$$

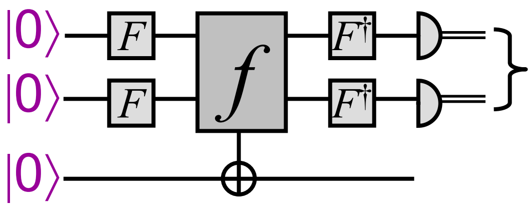
where $\omega = e^{2\pi i/m}$ (for n qubits, $m = 2^n$)

This is unitary and $F_2 = H$, the Hadamard transform

This generalization of H is an important component of several interesting quantum algorithms …

Quantum algorithm for Simon mod *m* (1)

$f(x_1, x_2) = f(y_1, y_2)$ iff $(x_1, x_2) - (y_1, y_2) \equiv k(r, 1) \pmod{m}$



turns out that the result is a random (s_1, s_2) such that $(s_1, s_2)\cdot(r, 1) \equiv 0 \pmod{m}$

The state right after the query is $\frac{1}{m}\sum_{x_1 \in \mathbb{Z}_m}\sum_{x_1 \in \mathbb{Z}_m} |x_1\rangle|x_2\rangle|f(x_1, x_2)\rangle$

Now, if the third register is measured in the computational basis then it collapses to some value, and state of the first two registers is a superposition of all (x_1, x_2) that f maps to that value, which is a state of the form

$$\frac{1}{\sqrt{m}}\sum_{k \in \mathbb{Z}_m} |x_1 + kr_1\rangle|x_2 + kr_2\rangle$$

$$= \frac{1}{\sqrt{m}}\Big(|(x_1, x_2)\rangle + |(x_1, x_2) + (r_1, r_2)\rangle + \cdots + |(x_1, x_2) + (m-1)(r_1, r_2)\rangle\Big)$$

Quantum algorithm for Simon mod m (2)

Here is the state again:

$$\frac{1}{\sqrt{m}} \sum_{k \in \mathbb{Z}_m} |x_1 + kr_1\rangle |x_2 + kr_2\rangle$$

$$= \frac{1}{\sqrt{m}} \Big(|(x_1, x_2)\rangle + |(x_1, x_2) + (r_1, r_2)\rangle + \cdots + |(x_1, x_2) + (m-1)(r_1, r_2)\rangle \Big)$$

The next step is to apply the two inverse Fourier transforms mod m, yielding

$$\left(F_m^\dagger \otimes F_m^\dagger\right) \frac{1}{\sqrt{m}} \sum_{k \in \mathbb{Z}_m} |x_1 + kr_1\rangle |x_2 + kr_2\rangle = \frac{1}{\sqrt{m}} \sum_{k \in \mathbb{Z}_m} F_m^\dagger |x_1 + kr_1\rangle F_m^\dagger |x_2 + kr_2\rangle$$

$$= \frac{1}{m^{3/2}} \sum_{k \in \mathbb{Z}_m} \sum_{s_1 \in \mathbb{Z}_m} \omega^{-s_1(x_1 + kr_1)} |s_1\rangle \sum_{s_2 \in \mathbb{Z}_m} \omega^{-s_2(x_2 + kr_2)} |s_2\rangle$$

$$= \frac{1}{\sqrt{m}} \sum_{s_1} \sum_{s_2} \left(\frac{1}{m} \sum_{k \in \mathbb{Z}_m} \omega^{-(s_1, s_2) \cdot ((x_1, x_2) + k(r_1, r_2))} \right) |s_1, s_2\rangle$$

$$= \frac{1}{\sqrt{m}} \sum_{s_1, s_2} \omega^{-(s_1, s_2) \cdot (x_1, x_2)} \left(\frac{1}{m} \sum_{k \in \mathbb{Z}_m} \omega^{-(s_1, s_2) \cdot (r_1, r_2) k} \right) |s_1, s_2\rangle$$

Quantum algorithm for Simon mod *m* (3)

Note that $$\frac{1}{m}\sum_{k\in\mathbb{Z}_m}\omega^{-(s_1,s_2)\cdot(r_1,r_2)k} = \begin{cases}1 & \text{if } (s_1,s_2)\cdot(r_1,r_2)=0\\ 0 & \text{otherwise}\end{cases}$$

So the amplitudes of all basis states $|s_1,s_2\rangle$ where $(s_1,s_2)\cdot(r_1,r_2)\neq 0$ are zero

Therefore, if the first two registers are measured, the result is a ***random*** (s_1,s_2) subject to the condition that it has dot product 0 with (r_1,r_2)

The dot product condition implies that (r_1,r_2) satisfies the linear relationship $s_1r_1+s_2r_2\equiv 0 \pmod{m}$

As with Simon's problem, we can repeat this process until we have enough linear relationships to deduce (r_1,r_2)

A complication is that, if the modulus m is not prime the we are not working over a *field*, so we are outside the framework of *linear algebra*

For the Discrete Log Problem, $m = p - 1$ (which is not prime) and $(r_1,r_2)=(1,r)$

Quantum algorithm for Simon mod m (4)

In the context of DLP, we have $(s_1, s_2) \cdot (r, 1) \equiv s_1 r + s_2 \equiv 0 \pmod{p-1}$

If s_1 has an inverse then we can solve for r as $r = -s_2/s_1$

In our $\bmod\ p-1$ arithmetic, if s_1 and $p-1$ are ***coprime*** (see below) then s_1 has an inverse $\bmod\ p-1$

Moreover, the probability that s_1 and $p-1$ are coprime occurs is not too small (and if it fails on one run then the algorithm can be run again)

Definition: a_1 and a_2 are ***coprime*** if their largest common divisor is 1 (for example, 12 land 21 are ***not*** coprime, since 3 is a common divisor, but 10 and 21 are coprime)

Lemma: if a_1 and a_2 are coprime then a_1 has an inverse modulo a_2

Proof idea: the Extended Euclidean Algorithm implies that if a_1 and a_2 are coprime then there exist integers b_1 and b_2 such that $b_1 a_1 + b_2 a_2 = 1$
(e.g., for 10 and 21, we have $(-2)10 + (1)21 = 1$)

This implies that $b_1 a_1 = 1 - b_2 a_2$ so $b_1 a_1 \equiv 1 \pmod{a_2}$

Therefore $b_1 = a_1^{-1} \bmod a_2$

Quantum algorithm for Simon mod m (5)

Steps that have been shown to be efficiently implementable (i.e., in terms of a number of 1- and 2-qubit/bit gates that scales polynomially with respect to the number of bits of m):

- Implementation of reversible gate for f
- The classical post-processing at the end

What's missing?

Implementation of the QFT f modulo m ($= p - 1$ for DLP)

Here, we'll just show how to implement the QFT for $m = 2^n$

Shor did this too, and showed that if the modulus is within a factor of 2 from $p - 1$, by using careful error-analysis, this was good enough, though the calculations and analysis become more complicated (we omit the details of this)

Continuing with the QFT for $m = 2^n$

Quantum Fourier transform

$$F_m = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{m-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(m-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \dots & \omega^{3(m-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{m-1} & \omega^{2(m-1)} & \omega^{3(m-1)} & \cdots & \omega^{(m-1)^2} \end{bmatrix}$$

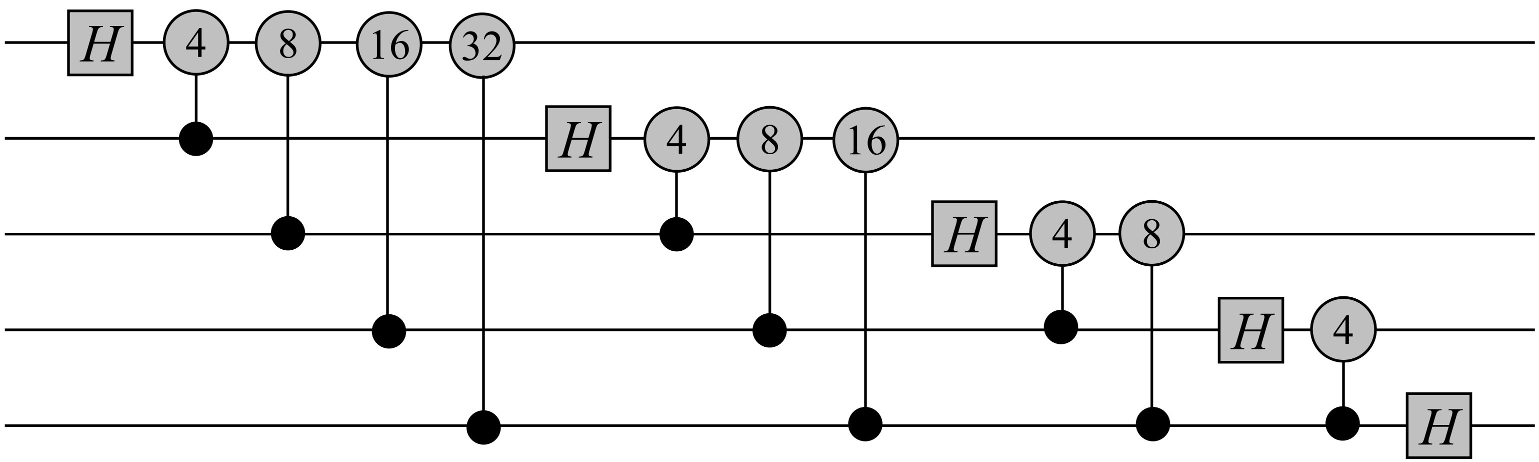
where $\omega = e^{2\pi i/m}$ (for n qubits, $m = 2^n$)

This is unitary and $F_2 = H$, the Hadamard transform

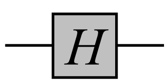
This generalization of H is an important component of several interesting quantum algorithms …

Computing the QFT for $m = 2^n$ (1)

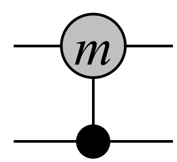
Quantum circuit for F_{32}:



and reverse order of qubits

Gates: 

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$



$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i/m} \end{bmatrix}$

For F_{2n} costs $O(n^2)$ gates

Computing the QFT for $m = 2^n$ (2)

Binary numbers (base-two representation of integers)

We identify $\{000, 001, 010, 011, 100, 101, 110, 111\} = \{0, 1, 2, 3, 4, 5, 6, 7\}$

Formally, for $a = a_1 a_2 \ldots a_n$, define $(a_1 a_2 \ldots a_n)$ to be the corresponding integer

Binary fractions (base-two representation of rational numbers)

What are (0.1)?, (0.01), (0.11)?
As in the base-ten case, shifting the radix point left by is equivalent to dividing by the base number
Therefore, $(0.1) = ½(1.0) = ½$, $(0.11) = ¼(11.0) = ¼(3) = ¾$ (etc)

Some expressions involving binary fractions

$e^{2\pi i (0.0)} = 1$, $e^{2\pi i (0.1)} = -1$

$e^{2\pi i (1.0)} = 1$, $e^{2\pi i (1.1)} = -1$

$e^{2\pi i (0.01)} = i$, $e^{2\pi i (0.11)} = -i$

Computing the QFT for $m = 2^n$ (3)

One way on seeing why this circuit works is to show:

1. For all $a_1 a_2 \ldots a_n \in \{0,1\}^n$, on input state $|a_1 a_2 \ldots a_n\rangle$ the output of the circuit (before reversing the qubits) is

$$\left(|0\rangle + e^{2\pi i(0.a_1 a_2 \ldots a_n)}|1\rangle\right)\left(|0\rangle + e^{2\pi i(0.a_2 \ldots a_n)}|1\rangle\right)\ldots\left(|0\rangle + e^{2\pi i(0.a_n)}|1\rangle\right)$$

2. And then

$$\left(|0\rangle + e^{2\pi i(0.a_n)}|1\rangle\right)\ldots\left(|0\rangle + e^{2\pi i(0.a_2 \ldots a_n)}|1\rangle\right)\left(|0\rangle + e^{2\pi i(0.a_1 a_2 \ldots a_n)}|1\rangle\right)$$

$$= \left(|0\rangle + \omega^{2^{n-1}(a)}|1\rangle\right)\ldots\left(|0\rangle + \omega^{2(a)}|1\rangle\right)\left(|0\rangle + \omega^{(a)}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} \omega^{(a)k}|k\rangle \qquad \left(\text{where } \omega = e^{2\pi i/2^n}\right)$$

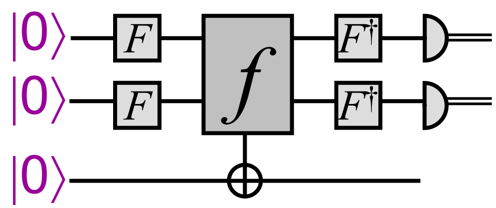
$$= F_{2^n}|a_1 a_2 \ldots a_n\rangle$$

Exercise: show these two steps in detail

Summary of DLP algorithm

Implement $f(x) = g^{x_1} a^{-x_2} \bmod p$ reversibly and F_{2^n} where $2^{n-1} < p - 1 < 2^n$

Execute this circuit:



If the measured results are s_1 and s_2 where s_1 and $p - 1$ are coprime then output $r = -s_2/s_1 \bmod p - 1$
(otherwise, execute above circuit again)

Hidden Subgroup Problem framework

Aside: hidden subgroup problem (commutative version)

Let G be a known group and H be an unknown subgroup of G

Let $f: G \rightarrow T$ have the property $f(x) = f(y)$ iff $x - y \in H$ (i.e., x and y are in the same ***coset*** of H)

Problem: given a black-box for computing f, determine H

Example 1: $G = (\mathbb{Z}_2)^n$ (the additive group) and $H = \{0, r\}$

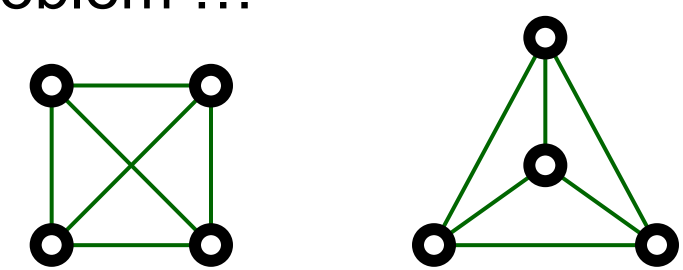
Example 2: $G = (\mathbb{Z}_{p-1})^2$ and $H = \{(0,0), (r,1), (2r,2), \ldots, ((p-2)r, p-2)\}$

Example 3: $G = \mathbb{Z}$ and $H = r\mathbb{Z}$ (Shor's factoring algorithm was originally approached this way. A complication that arises is that $\mathbb{Z}$ is infinite. We'll use a different approach)

Aside: hidden subgroup problem (noncommutative version)

Example 4: $G = S_n$ (the symmetric group, consisting of all permutations on n objects—which is not commutative) and H is any subgroup of G (and we use ***left*** cosets throughout)

A quantum algorithm for this instance of HSP ***would*** lead to an efficient quantum algorithm for the graph isomorphism problem …



… ***alas*** no efficient quantum has been found for this instance of HSP, despite significant effort by many people

Eigenvalue estimation problem (a.k.a. phase estimation)

Note: this will lead to a factoring algorithm similar to Shor's

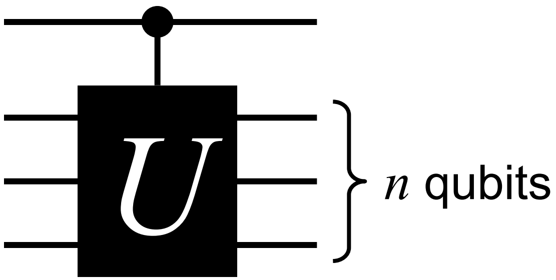
A simplified example

U is an unknown unitary operation on n qubits

$|\psi\rangle$ is an eigenvector of U, with eigenvalue $\lambda = +1$ or -1

Input: a black-box for a controlled-U

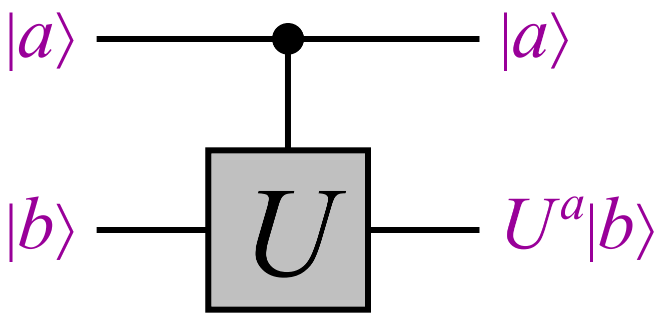
and a copy of the state $|\psi\rangle$



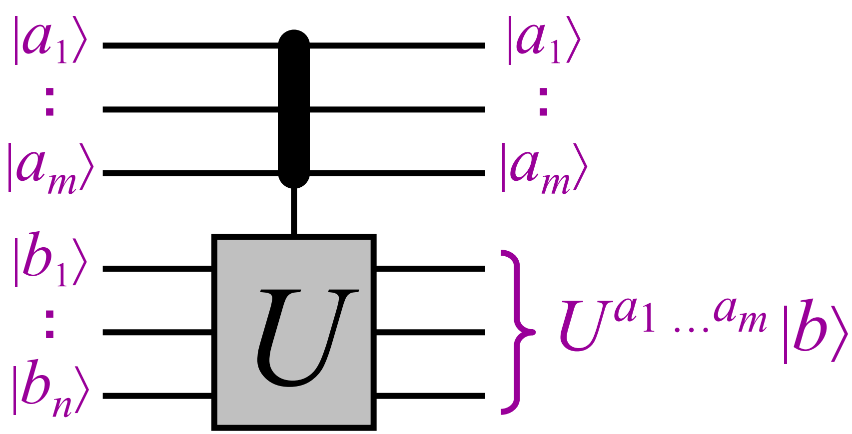
Output: the eigenvalue λ

Exercise: solve this making a single query to the controlled-U

Generalized controlled-U gates



$$\begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix}$$



$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & U & 0 & \cdots & 0 \\ 0 & 0 & U^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & U^{2^m-1} \end{bmatrix}$$

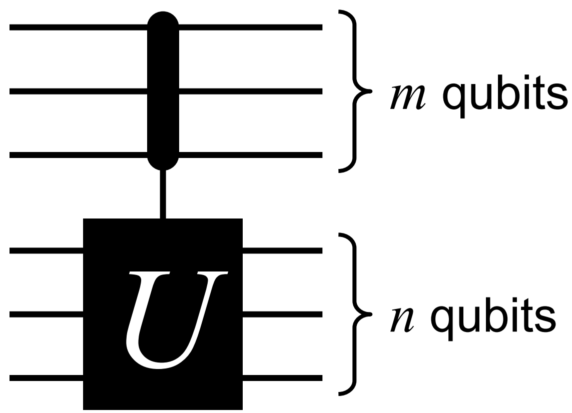
Example: $|1101\rangle|0101\rangle \mapsto |1101\rangle U^{1101}|0101\rangle$

Eigenvalue estimation problem

U is a unitary operation on n qubits

$|\psi\rangle$ is an eigenvector of U, with eigenvalue $e^{2\pi i\phi}$
$(0 \leq \phi < 1)$

Input: black-box for

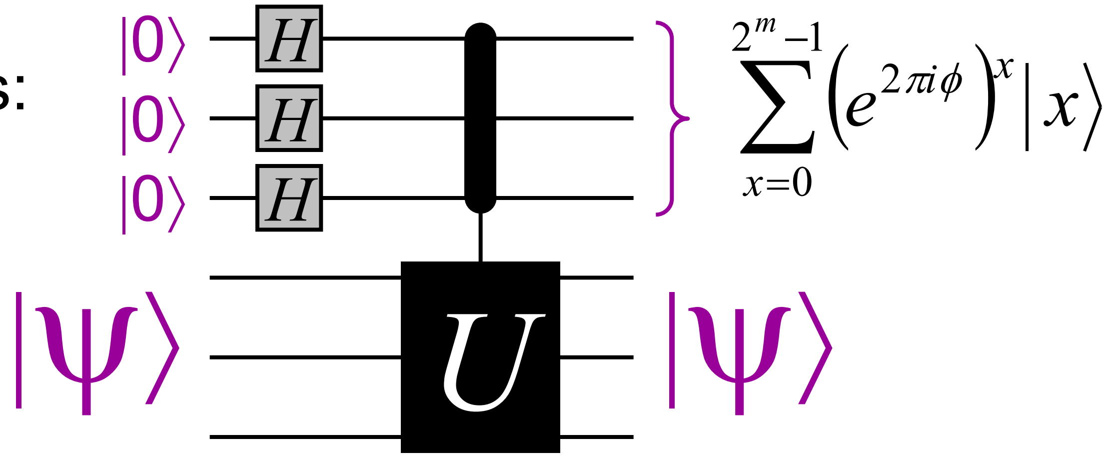


and a copy of $|\psi\rangle$

Output: ϕ (m-bit approximation)

Algorithm for eigenvalue estimation (1)

Starts off as:



$|00\ldots0\rangle|\psi\rangle$

$|a\rangle|b\rangle \rightarrow |a\rangle U^a|b\rangle$

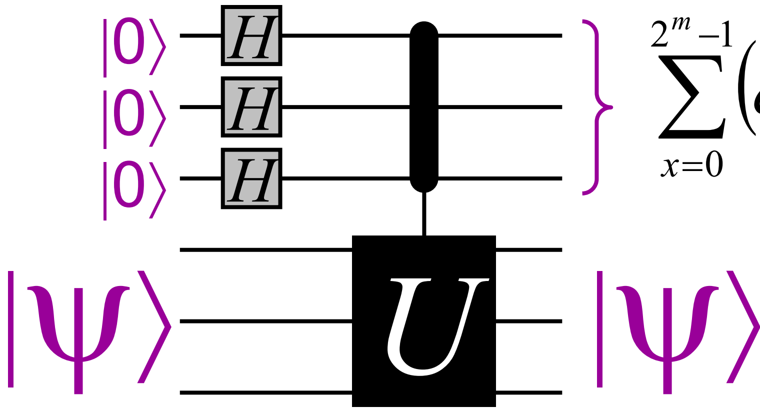
$\mapsto (|0\rangle+|1\rangle)(|0\rangle+|1\rangle)\ldots(|0\rangle+|1\rangle)|\psi\rangle$

$= (|000\rangle+|001\rangle+|010\rangle+|011\rangle+\ldots+|111\rangle)|\psi\rangle$

$= (|0\rangle+|1\rangle+|2\rangle+|3\rangle+\ldots+|2^m-1\rangle)|\psi\rangle$

$\mapsto (|0\rangle+e^{2\pi i\phi}|1\rangle+(e^{2\pi i\phi})^2|2\rangle+(e^{2\pi i\phi})^3|3\rangle+\ldots+(e^{2\pi i\phi})^{2^m-1}|2^m-1\rangle)|\psi\rangle$

Algorithm for eigenvalue estimation (2)



$$\sum_{x=0}^{2^m-1} \left(e^{2\pi i \phi}\right)^x |x\rangle$$

Recall that $F_M \left| a_1 a_2 \ldots a_m \right\rangle = \sum_{x=0}^{2^m-1} \left(e^{2\pi i (0.a_1 a_2 \ldots a_m)}\right)^x |x\rangle$

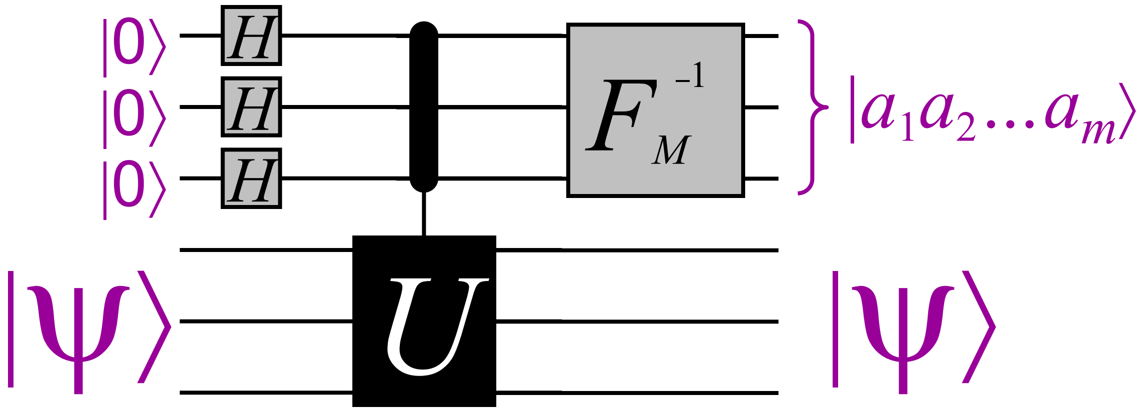
$$F_M^{-1} = \frac{1}{\sqrt{M}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(M-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(M-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(M-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(M-1)} & \omega^{-2(M-1)} & \omega^{-3(M-1)} & \cdots & \omega^{-(M-1)^2} \end{bmatrix}$$

Therefore, when

$\phi = 0.a_1 a_2 \ldots a_m$

applying the ***inverse*** of F_M yields ϕ (digits)

Algorithm for eigenvalue estimation (3)



If $\phi = 0.a_1 a_2 \ldots a_m$ then the above procedure yields $|a_1 a_2 \ldots a_m\rangle$ (from which ϕ can be deduced exactly)

But what ϕ if is not of this nice form?

Example: $\phi = ⅓ = 0.010101010101010101\ldots$

Algorithm for eigenvalue estimation (4)

What if ϕ is not of the nice form $\phi = 0.a_1 a_2 \ldots a_m$?

Example: $\phi = ⅓ = 0.\underline{010101010}1010101\ldots$

Let's calculate what the previously-described procedure does:

Let $a/2^m = 0.a_1 a_2 \ldots a_m$ be an m-bit approximation of ϕ,
in the sense that $\phi = a/2^m + \delta$, where $|\delta| \le 1/2^{m+1}$

$$(F_M)^{-1} \sum_{x=0}^{2^m-1} \left(e^{2\pi i\phi}\right)^x |x\rangle = \frac{1}{2^m} \sum_{y=0}^{2^m-1} \sum_{x=0}^{2^m-1} e^{-2\pi ixy/2^m} e^{2\pi i\phi x} |y\rangle$$

$$= \frac{1}{2^m} \sum_{y=0}^{2^m-1} \sum_{x=0}^{2^m-1} e^{-2\pi ixy/2^m} e^{2\pi i\left(\frac{a}{2^m}+\delta\right)x} |y\rangle$$

$$= \frac{1}{2^m} \sum_{y=0}^{2^m-1} \sum_{x=0}^{2^m-1} e^{2\pi i(a-y)x/2^m} e^{2\pi i\delta x} |y\rangle$$

What is the amplitude of $|a_1 a_2 \ldots a_m\rangle$?

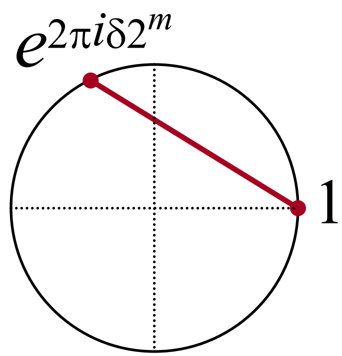
Algorithm for eigenvalue estimation (5)

State is: $$\frac{1}{2^m}\sum_{y=0}^{2^m-1}\sum_{x=0}^{2^m-1} e^{2\pi i(a-y)x/2^m} e^{2\pi i\delta x}|y\rangle$$

geometric series!

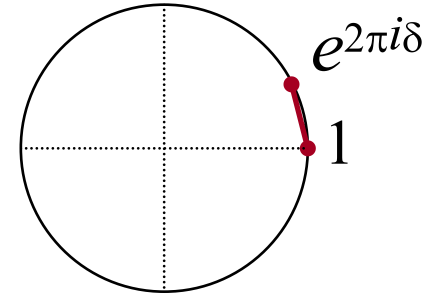
The amplitude of $|y\rangle$, for $y = a$ is $\frac{1}{2^m}\sum_{x=0}^{2^m-1} e^{2\pi i\delta x} = \frac{1}{2^m}\frac{1-\left(e^{2\pi i\delta}\right)^{2^m}}{1-e^{2\pi i\delta}}$

Numerator:



lower bounded by
$2\pi\delta 2^m(2/\pi) > 4\delta 2^m$

Denominator:



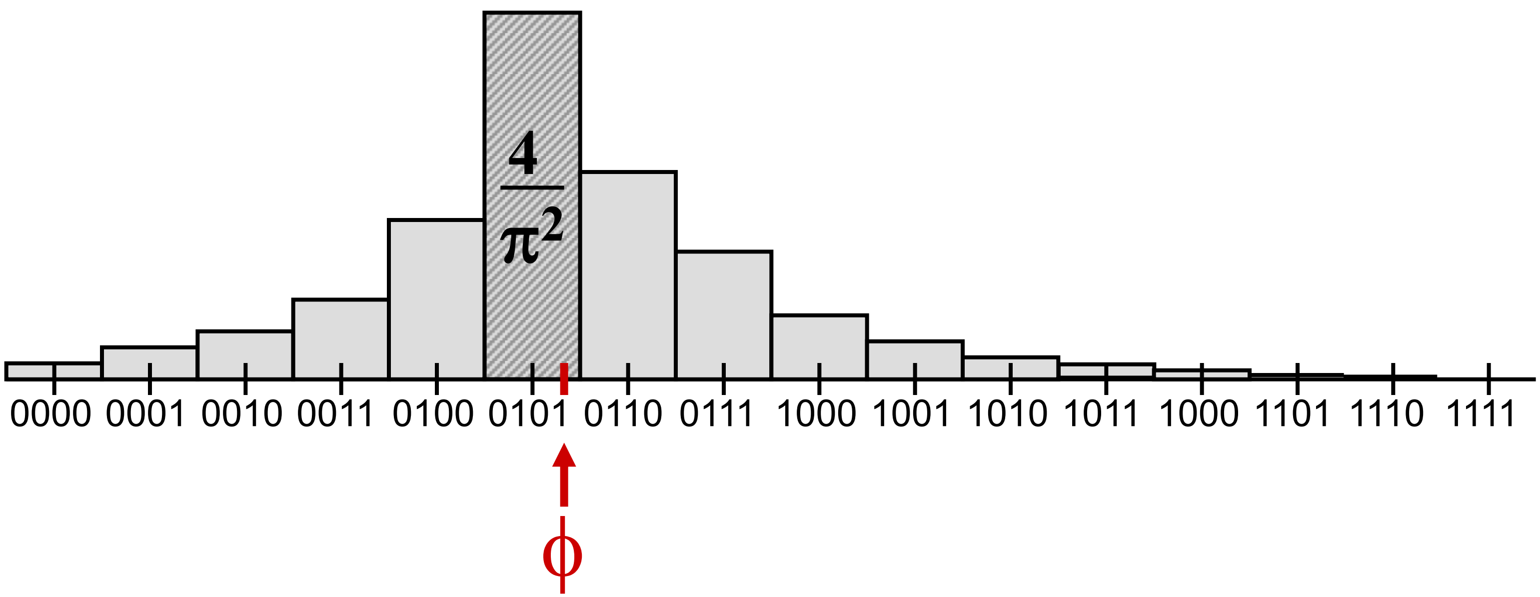
upper bounded by $2\pi\delta$

Therefore, the absolute value of the amplitude of $|y\rangle$ is at least the quotient of $(1/2^m)$(numerator/denominator), which is $2/\pi$

Algorithm for eigenvalue estimation (6)

Therefore, the probability of measuring an m-bit approximation of ϕ is always at least $4/\pi^2 \approx 0.4$

For example, when $\phi = ⅓ = 0.\underline{0101}0101010101\ldots$, the outcome probabilities look roughly like this:



Note: with **$2m$-qubit** control gate, error probability is exponentially small