Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lecture 21 (2019)

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Schmidt decomposition

Schmidt decomposition

Theorem:

Let $|\psi\rangle$ be ***any*** bipartite quantum state:

$$|\psi\rangle = \sum_{a=1}^{m}\sum_{b=1}^{n}\alpha_{a,b}|a\rangle\otimes|b\rangle$$ (where we can assume $n \leq m$)

Then there exist orthonormal states $|\mu_1\rangle, |\mu_2\rangle, \ldots, |\mu_n\rangle$ and $|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_n\rangle$ such that

- $|\psi\rangle = \sum_{c=1}^{n}\sqrt{p_c}\,|\mu_c\rangle\otimes|\varphi_c\rangle$
- $|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_n\rangle$ are the eigenvectors of $\mathrm{Tr}_1|\psi\rangle\langle\psi|$

Schmidt decomposition: proof (1)

The density matrix for state $|\psi\rangle$ is given by $|\psi\rangle\langle\psi|$

Tracing out the first system, we obtain the density matrix of the second system, $\rho = \mathrm{Tr}_1|\psi\rangle\langle\psi|$

Since ρ is a density matrix, we can express $\rho = \sum_{c=1}^{n} p_c |\varphi_c\rangle\langle\varphi_c|$,

where $|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_n\rangle$ are orthonormal eigenvectors of ρ

Now, returning to $|\psi\rangle$, we can express $|\psi\rangle = \sum_{c=1}^{n} |\nu_c\rangle \otimes |\varphi_c\rangle$, where $|\nu_1\rangle, |\nu_2\rangle, \ldots, |\nu_n\rangle$ are ***just some arbitrary vectors*** (not necessarily valid quantum states; for example, they might not have unit length, and we cannot presume they're orthogonal)

Schmidt decomposition: proof (2)

Claim: $\langle \nu_c | \nu_{c'} \rangle = \begin{cases} p_c & \text{if } c = c' \\ 0 & \text{if } c \neq c' \end{cases}$

Proof of Claim: Compute the partial trace Tr_1 of $|\psi\rangle\langle\psi|$ from

$$|\psi\rangle\langle\psi| = \left(\sum_{c=1}^{n} |\nu_c\rangle \otimes |\varphi_c\rangle\right)\left(\sum_{c'=1}^{n} \langle \nu_{c'}| \otimes \langle \varphi_{c'}|\right) = \sum_{c=1}^{n}\sum_{c'=1}^{n} |\nu_c\rangle\langle\nu_{c'}| \otimes |\varphi_c\rangle\langle\varphi_{c'}|$$

Note that: $\mathrm{Tr}_1(A \otimes B) = \mathrm{Tr}(A) \cdot B$	Example: $\mathrm{Tr}_1(\rho \otimes \sigma) = \sigma$

$$\mathrm{Tr}_1\left(\sum_{c=1}^{n}\sum_{c'=1}^{n} |\nu_c\rangle\langle\nu_{c'}| \otimes |\varphi_c\rangle\langle\varphi_{c'}|\right) = \sum_{c=1}^{n}\sum_{c'=1}^{n} \mathrm{Tr}\left(|\nu_c\rangle\langle\nu_{c'}|\right)|\varphi_c\rangle\langle\varphi_{c'}| \quad \text{(linearity)}$$

$$= \sum_{c=1}^{n}\sum_{c'=1}^{n} \langle\nu_{c'}|\nu_c\rangle |\varphi_c\rangle\langle\varphi_{c'}|$$

Since $\sum_{c=1}^{n}\sum_{c'=1}^{n} \langle\nu_{c'}|\nu_c\rangle \otimes |\varphi_c\rangle\langle\varphi_{c'}| = \sum_{c=1}^{n} p_c |\varphi_c\rangle\langle\varphi_c|$ the claim follows

Schmidt decomposition: proof (3)

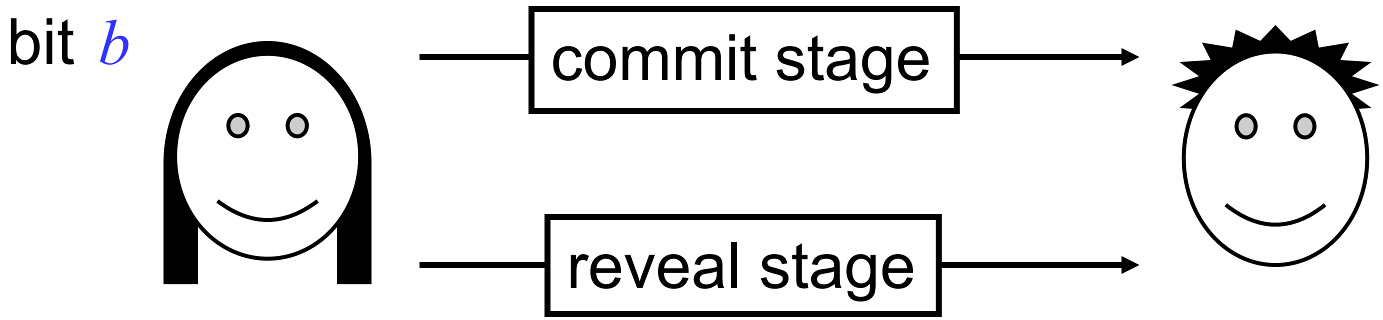
Normalize the $|\nu_c\rangle$ by setting $|\mu_c\rangle = \frac{1}{\sqrt{p_c}} |\nu_c\rangle$

Then $\langle\mu_c|\mu_{c'}\rangle = \begin{cases} 1 & \text{if } c = c' \\ 0 & \text{if } c \neq c' \end{cases}$

and $|\psi\rangle = \sum_{c=1}^{n} \sqrt{p_c} |\mu_c\rangle \otimes |\varphi_c\rangle$

The story of bit commitment

Bit-commitment



- Alice has a bit b that she wants to ***commit*** to Bob:
- After the ***commit*** stage, Bob should know nothing about b, but Alice should not be able to change her mind
- After the ***reveal*** stage, either:
  - Bob should learn b and accept its value, or
  - Bob should reject Alice's reveal message, if she deviates from the protocol

Simple physical implementation

- **Commit:** Alice writes b down on a piece of paper, locks it in a safe, sends the safe to Bob, but keeps the key
- **Reveal:** Alice sends the key to Bob, who then opens the safe
- Desirable properties:
  - **Binding:** Alice cannot change b after **commit**
  - **Concealing:** Bob learns nothing about b until **reveal**

Question: why should anyone care about bit-commitment?

Answer: it is a useful primitive operation for other protocols, such as coin-flipping, and "zero-knowledge proof systems"

Complexity-theoretic implementation

Based on a ***one-way function**** $f : \{0,1\}^n \rightarrow \{0,1\}^n$ and a ***hard-predicate*** $h : \{0,1\}^n \rightarrow \{0,1\}$ for f

Commit: Alice picks a random $x \in \{0,1\}^n$, sets $y = f(x)$ and $c = b \oplus h(x)$ and then sends y and c to Bob

Reveal: Alice sends x to Bob, who verifies that $y = f(x)$ and then sets $b = c \oplus h(x)$

This is (i) perfectly binding and (ii) computationally concealing, based on the hardness of predicate h

* should be one-to-one

Quantum implementation (1)

- Inspired by the success of QKD, one can try to use the properties of quantum mechanical systems to design an information-theoretically secure bit-commitment scheme
- One simple idea:
  - To **commit** to **0**, Alice sends a random sequence from $\{|0\rangle, |1\rangle\}$
  - To **commit** to **1**, Alice sends a random sequence from $\{|+\rangle, |-\rangle\}$
  - Bob measures each qubit received in a random basis
  - To **reveal**, Alice tells Bob exactly which states she sent in the commitment stage (by sending its index 00, 01, 10, or 11), and Bob checks for consistency with his measurement results

Intuition:

Typical commitment to **0**: $|0\rangle|1\rangle|1\rangle|0\rangle|0\rangle|1\rangle|0\rangle|1\rangle|0\rangle|0\rangle|0\rangle|1\rangle|0\rangle|1\rangle|1\rangle|0\rangle$

Typical commitment to **1**: $|-\rangle|-\rangle|+\rangle|-\rangle|+\rangle|+\rangle|+\rangle|-\rangle|+\rangle|+\rangle|-\rangle|+\rangle|-\rangle|-\rangle|+\rangle|-\rangle$

Quantum implementation (2)

A paper appeared in 1993 proposing a quantum bit-commitment scheme and a proof of security

Impossibility proof (I)

- Not only was the 1993 scheme shown to be insecure, but it was later shown that ***no such scheme can exist!***
- To understand the impossibility proof, recall the ***Schmidt decomposition:***

Let $|\psi\rangle$ be any bipartite quantum state:

$$|\psi\rangle = \sum_{a=1}^{n}\sum_{b=1}^{n}\alpha_{a,b}|a\rangle|b\rangle$$

Then there exist orthonormal states

$|\mu_1\rangle, |\mu_2\rangle, \ldots, |\mu_n\rangle$ and $|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_n\rangle$ such that

$$|\psi\rangle = \sum_{c=1}^{n}\beta_c|\mu_c\rangle|\phi_c\rangle$$

Eigenvectors of $\mathrm{Tr}_1|\psi\rangle\langle\psi|$

[Mayers '96][Lo & Chau '96]

Impossibility proof (II)

- **Corollary:** if $|\psi_0\rangle$, $|\psi_1\rangle$ are two bipartite states such that $\mathrm{Tr}_1|\psi_0\rangle\langle\psi_0| = \mathrm{Tr}_1|\psi_1\rangle\langle\psi_1|$ then there exists a unitary U (acting on the first register) such that $(U \otimes I)|\psi_0\rangle = |\psi_1\rangle$
- **Proof:**

$$|\psi_0\rangle = \sum_{c=1}^{n} \beta_c |\mu_c\rangle|\phi_c\rangle \quad \text{and} \quad |\psi_1\rangle = \sum_{c=1}^{n} \beta_c |\mu'_c\rangle|\phi_c\rangle$$

We can define U so that $U|\mu_c\rangle = |\mu'_c\rangle$ for $c = 1,2,\ldots,n$ ∎

- Protocol can be "purified" so that Alice's commit states are $|\psi_0\rangle$ & $|\psi_1\rangle$ (where she sends the second register to Bob)

- By applying U to her register, Alice can change her commitment from $b = 0$ to $b = 1$ (by changing $|\psi_0\rangle$ to $|\psi_1\rangle$)

Separable states

(very briefly)

(brief)

Separable states

A bipartite (i.e. two register) state ρ is a:

- ***product state*** if $\rho = \sigma \otimes \xi$
- ***separable state*** if $\rho = \sum_{j=1}^{m} p_j \sigma_j \otimes \xi_j$ $\quad (p_1, \ldots, p_m \geq 0)$

  (i.e. a probabilistic mixture of product states)
- ***entangled*** = not separable

Since mixed states might be expressible as a mixture in several different ways, determining whether they are separable is tricky

Question: which of the following states are separable?

$$\rho_1 = \tfrac{1}{2}\big(|00\rangle + |11\rangle\big)\big(\langle 00| + \langle 11|\big)$$

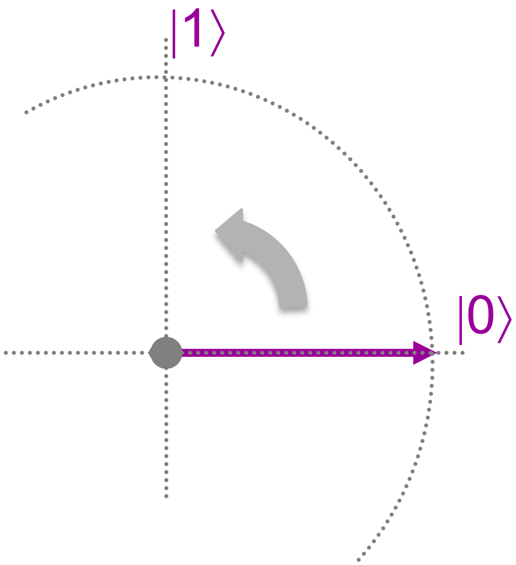
$$\rho_2 = \tfrac{1}{2}\big(|00\rangle + |11\rangle\big)\big(\langle 00| + \langle 11|\big) + \tfrac{1}{2}\big(|00\rangle - |11\rangle\big)\big(\langle 00| - \langle 11|\big)$$

Continuous-time evolution

(very briefly)

Continuous-time evolution

Although we've expressed quantum operations in discrete terms, in real physical systems, the evolution is continuous



Let H be any ***Hermitian*** matrix and $t \in \mathbf{R}$

Then e^{iHt} is ***unitary*** — why?

H is called a ***Hamiltonian***

$$H = U^\dagger D U, \text{ where } D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$$

$$e^{iHt} = U^\dagger e^{iDt} U = U^\dagger \begin{pmatrix} e^{i\lambda_1 t} & & \\ & \ddots & \\ & & e^{i\lambda_d t} \end{pmatrix} U \quad \text{(unitary)}$$