Introduction to Quantum Information Processing
QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lectures 12 (2019)

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Recall: quantum channels

Also known as: **general quantum operations**, **admissible operations**, **completely positive trace preserving maps**

Let $A_1, A_2, \ldots, A_m$ be any matrices satisfying $\sum_{j=1}^{m} A_j^\dagger A_j = I$

Then the mapping $\rho \mapsto \sum_{j=1}^{m} A_j \rho A_j^\dagger$ is a quantum channel

<u>Examples of channels:</u>

Unitary operation: $m=1$ and $A_1 = U$ (ρ maps to $U\rho U^\dagger$)

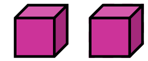
Probabilistic mixture of unitary operations: $A_k = \sqrt{p_k} U_k$, where $(p_1, \ldots, p_m)$ is a probability distribution and each U_k is unitary

Decoherence: $A_0 = |\phi_0\rangle\langle\phi_0|$, $A_1 = |\phi_1\rangle\langle\phi_1|$, …, $A_{d-1} = |\phi_{d-1}\rangle\langle\phi_{d-1}|$
(equivalent to measuring in in some basis and outputting the collapsed state)

Partial trace (of a bipartite register):
$A_0 = I\otimes\langle 0|$, $A_2 = I\otimes\langle 1|$, …, $A_{d-1} = I\otimes\langle d-1|$

More about the partial trace (1)

Intuition: for a bipartite system in state ρ, $\mathrm{Tr}_2(\rho)$ is the state of the 1st register (if the 2nd register is discarded)



Easy case: for a product state $\rho = \sigma \otimes \mu$, it holds that $\mathrm{Tr}_2(\rho) = \sigma$

In general, $\mathrm{Tr}_2(\rho)$ is not so trivial, since the state of a two-register system may not be of the form $\sigma \otimes \mu$ (it may contain ***entanglement*** or ***correlations***)

Aside 1

The "full" trace, is $\mathrm{Tr}(\rho) = \langle \phi_0 | \rho | \phi_0 \rangle + \langle \phi_1 | \rho | \phi_1 \rangle + \ldots + \langle \phi_{d-1} | \rho | \phi_{d-1} \rangle$, where $|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{d-1}\rangle$ is any orthonormal basis (equivalently, for square matrix M, $\mathrm{Tr}(M)$ is the sum of the diagonal entries of M)

Note: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ and $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$ hold

… but, in general, $\mathrm{Tr}(ABC) \neq \mathrm{Tr}(ACB)$ and $\mathrm{Tr}(AB) \neq \mathrm{Tr}(A)\,\mathrm{Tr}(B)$

Aside 2

For any matrices A, B, C, D where the dimensions are compatible,
$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

More about the partial trace (2)

1. Definition of $\mathrm{Tr}_2(\ )$ in terms of measurements

Let $|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{d-1}\rangle$ be an orthonormal basis for the second register

Imagine measuring the 2nd register with respect to the above basis and discarding the second register

If the state happens to be ***pure*** $|\psi\rangle = \sum_{j=0}^{d-1} \lambda_j |\nu_j\rangle \otimes |\phi_j\rangle$ then the outcome is $|\nu_j\rangle$ with probability $|\lambda_j|^2$ for each $j \in \{0,1, \ldots, d-1\}$

This is exactly what is produced by the channel defined by the operators

$A_0 = I \otimes \langle\phi_0|$, $A_1 = I \otimes \langle\phi_1|$, …, $A_{d-1} = I \otimes \langle\phi_{d-1}|$

These operators also represent the measurement in the case where the state is ***mixed*** (by considering probability distributions over pure states)

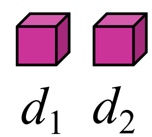
Based on this, we can define $\mathrm{Tr}(\rho) = \sum_{k=0}^{d-1} (I \otimes \langle\phi_k|)\, \rho\, (I \otimes |\phi_k\rangle)$

More about the partial trace (3)

2. Alternate definition of $\mathrm{Tr}_2(\ )$ in terms of a linear extension

The partial trace $\mathrm{Tr}_2(\rho)$, can also be defined as the unique linear operator* satisfying the identity $\mathrm{Tr}_2(\sigma \otimes \mu) = \sigma$

* By ***linear operator***, we mean an operator that maps $d_1 d_2$ x $d_1 d_2$ matrices to d_1 x d_1 matrices (where d_1 and d_2 are the dimensions of the two registers) such that



$$F(\alpha A + \beta B) = \alpha F(A) + \beta F(B)$$

for all $A, B \in \mathbb{C}^{d_1 d_2 \times d_1 d_2}$ and $\alpha, \beta \in \mathbb{C}$

More about the partial trace (4)

3. Explicit expression for $\mathrm{Tr}_2(\;)$

$$\mathrm{Tr}_2\begin{bmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{bmatrix} = \begin{bmatrix} \rho_{00,00}+\rho_{01,01} & \rho_{00,10}+\rho_{01,11} \\ \rho_{10,00}+\rho_{11,01} & \rho_{10,10}+\rho_{11,11} \end{bmatrix}$$

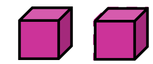
and

$$\mathrm{Tr}_1\begin{bmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{01,00} & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{10,00} & \rho_{10,01} & \rho_{10,10} & \rho_{10,11} \\ \rho_{11,00} & \rho_{11,01} & \rho_{11,10} & \rho_{11,11} \end{bmatrix} = \begin{bmatrix} \rho_{00,00}+\rho_{10,10} & \rho_{00,01}+\rho_{10,11} \\ \rho_{01,00}+\rho_{11,10} & \rho_{01,01}+\rho_{11,11} \end{bmatrix}$$

Channel for *adding* an extra register

Adding an extra register that is in some fixed state (say $|0\rangle\langle 0|$)

(it's kind of complementary to the partial trace)



$|0\rangle\langle 0|$

A channel with just one operator $A_0 = I \otimes |0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

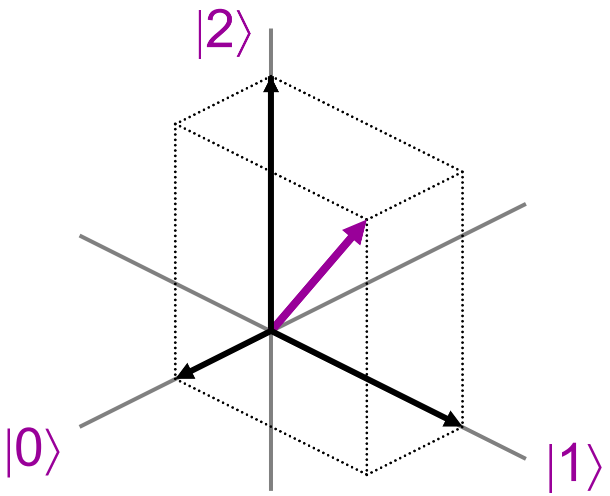
Any state of the form ρ becomes $\rho \otimes |0\rangle\langle 0|$

More generally, to add a register in state $|\phi\rangle$, use the operator $A_0 = I \otimes |\phi\rangle$

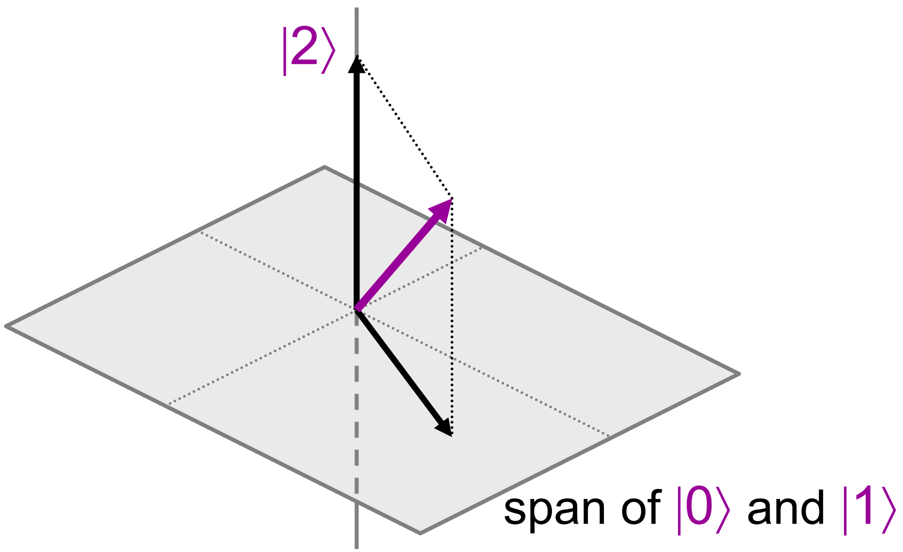
Exercise: what is the channel corresponding to adding a register in a ***mixed*** state σ?

General quantum measurements

Prelude: projective measurements



$$P_0 = |0\rangle\langle 0|$$
$$P_1 = |1\rangle\langle 1|$$
$$P_2 = |2\rangle\langle 2|$$



$$P_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$$
$$P_2 = |2\rangle\langle 2|$$

In both cases, there is a complete set of mutually orthogonal projectors:

$\sum_j P_j = I$ and $P_i P_j = 0$

The probability of outcome j is $\langle\psi|P_j^\dagger P_j|\psi\rangle = \mathrm{Tr}\big(|\psi\rangle\langle\psi|P_j^\dagger P_j\big)$ using $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$

The collapsed state is the projected vector, but normalized

Generalized measurements (1)

Let $A_1, A_2, \ldots, A_m$ be any matrices satisfying $\sum_{j=1}^{m} A_j^\dagger A_j = I$

Corresponding ***generalized measurement*** is a stochastic operation on ρ that, with probability $\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)$, produces outcome:

$$\begin{cases} \boldsymbol{j} & \text{(classical information)} \\ \dfrac{A_j \rho A_j^\dagger}{\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)} & \text{(the collapsed quantum state)} \end{cases}$$

Example 1: $A_j = |\phi_j\rangle\langle\phi_j|$ (rank-1 orthogonal projectors)

Can calculate that this is consistent with previous definitions (see next slide)

Question: what if we do the above but don't look at $\boldsymbol{j}$?

Answer: we get the channel corresponding to $A_1, A_2, \ldots, A_m$

Generalized measurements (2)

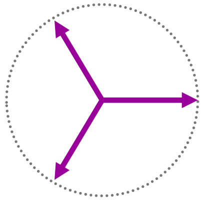
When $A_j = |\phi_j\rangle\langle\phi_j|$ are orthogonal projectors and $\rho = |\psi\rangle\langle\psi|$,

$$
\begin{aligned}
\mathrm{Tr}\left(A_j \rho A_j^\dagger\right) &= \mathrm{Tr}|\phi_j\rangle\langle\phi_j|\psi\rangle\langle\psi|\phi_j\rangle\langle\phi_j| \\
&= \langle\phi_j|\psi\rangle\langle\psi|\phi_j\rangle\langle\phi_j|\phi_j\rangle \\
&= |\langle\phi_j|\psi\rangle|^2
\end{aligned}
$$

Moreover, $$\frac{A_j \rho A_j^\dagger}{\mathrm{Tr}\left(A_j \rho A_j^\dagger\right)} = \frac{|\varphi_j\rangle\langle\varphi_j|\psi\rangle\langle\psi|\varphi_j\rangle\langle\varphi_j|}{\left|\langle\varphi_j|\psi\rangle\right|^2} = |\varphi_j\rangle\langle\varphi_j|$$

Generalized measurements (3)

Example 3 (trine state "measurent"):



Let $|\varphi_0\rangle = |0\rangle$, $|\varphi_1\rangle = -1/2|0\rangle + \sqrt{3}/2|1\rangle$, $|\varphi_2\rangle = -1/2|0\rangle - \sqrt{3}/2|1\rangle$

Define $A_0 = \sqrt{2/3}|\varphi_0\rangle\langle\varphi_0| = \sqrt{\frac{2}{3}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A_1 = \sqrt{2/3}|\varphi_1\rangle\langle\varphi_1| = \frac{1}{4}\begin{bmatrix} \sqrt{2/3} & +\sqrt{2} \\ +\sqrt{2} & \sqrt{6} \end{bmatrix}$ $A_2 = \sqrt{2/3}|\varphi_2\rangle\langle\varphi_2| = \frac{1}{4}\begin{bmatrix} \sqrt{2/3} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{6} \end{bmatrix}$

Then $A_0^\dagger A_0 + A_1^\dagger A_1 + A_2^\dagger A_2 = I$

If the input itself is an unknown trine state, $|\varphi_k\rangle\langle\varphi_k|$, then the probability that classical outcome is k is 2/3 = 0.6666…

POVM measurements

(POVM = **P**ositive **O**perator **V**alued **M**easure)

Often generalized measurements arise in contexts where we only care about the classical part of the outcome (not the residual quantum state), and then the definition can be simplified as follows

The probability of outcome ***j*** is $\mathrm{Tr}\left(A_j \rho A_j^\dagger\right) = \mathrm{Tr}\left(\rho A_j^\dagger A_j\right)$

POVM measurements:

Let $E_1, E_2, \ldots, E_m$ be positive semidefinite and with $\sum_{j=1}^{m} E_j = I$

The probability of outcome ***j*** is $\mathrm{Tr}\left(\rho E_j\right)$

Note: for a POVM measurement, there is no well-defined residual state, because the corresponding $A_1, A_2, \ldots, A_m$ are not uniquely defined

“Mother of all operations”

Let $A_{1,1}, A_{1,2}, \ldots, A_{1,m_1}$ satisfy $\sum_{j=1}^{k}\sum_{i=1}^{m_j} A_{j,i}^\dagger A_{j,i} = I$

$A_{2,1}, A_{2,2}, \ldots, A_{2,m_2}$

$A_{k,1}, A_{k,2}, \ldots, A_{k,m_k}$

Then there is a quantum operation that, on input ρ, produces

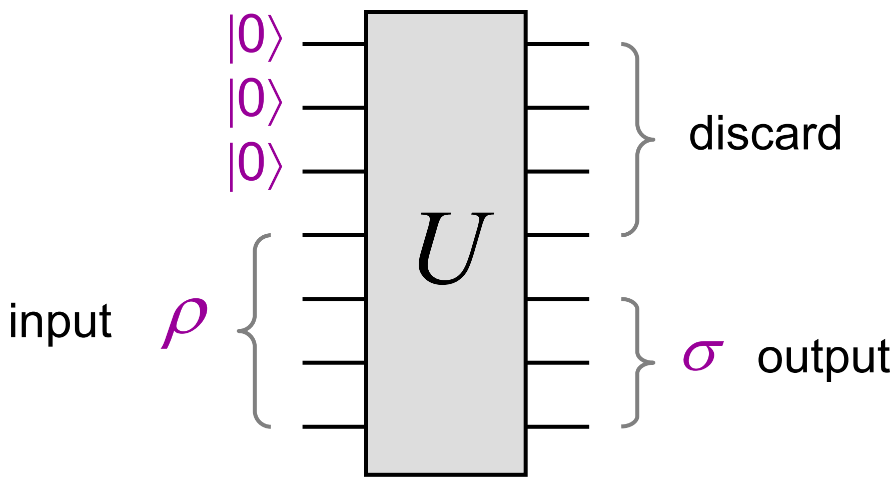
with probability $\sum_{i=1}^{m_j} \mathrm{Tr}\left(A_{j,i}\rho A_{j,i}^\dagger\right)$ the state:

$$\begin{cases} j & \text{(classical information)} \\ \dfrac{\sum_{i=1}^{m_j} A_{j,i}\rho A_{j,i}^\dagger}{\sum_{i=1}^{m_j} \mathrm{Tr}\left(A_{j,i}\rho A_{j,i}^\dagger\right)} & \text{(the collapsed quantum state)} \end{cases}$$

Simulations among operations

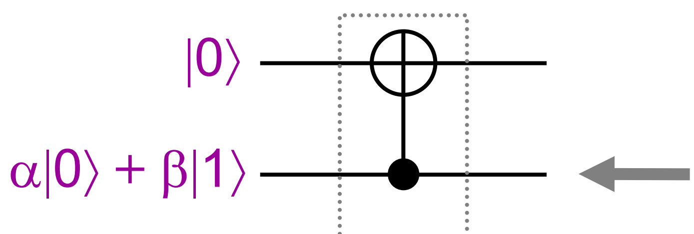
Simulations among operations (1)

Theorem 1: any ***quantum channel*** can be simulated by applying a unitary operation on a larger quantum system:



This specification of a quantum channel is called the ***Stinespring*** form

Example: decoherence



$$\rho = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}$$

Simulations among operations (2)

Proof of Theorem 1:

Let $A_1, A_2, \ldots, A_{2^k}$ be any 2^m x 2^n matrices such that

Kraus operators

$$\sum_{j=1}^{2^k} A_j^\dagger A_j = I$$

This defines a mapping from n qubits to m qubits

$$\rho \mapsto \sum_{j=1}^{2^k} A_j \rho A_j^\dagger$$

This specification of the quantum operation is called the ***Kraus*** form

Simulations among operations (3)

Set $V = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{2^k} \end{bmatrix}$

Since $V^\dagger V =$

$$\begin{bmatrix} A_1^\dagger & A_2^\dagger & \cdots & A_{2^k}^\dagger \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{2^k} \end{bmatrix} = I$$

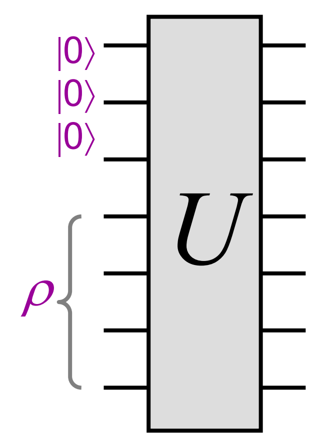
the columns of V are ***orthonormal***

Let U be any unitary matrix with first 2^n columns from V

$$U = [\, V \,\vdots\, W \,]$$

U is a 2^{m+k} x 2^{m+k} matrix (and its columns partition into 2^{m-n+k} blocks of size 2^n)

Now, consider the circuit:



Simulations among operations (4)

The output state of the circuit is $U(|00\cdots0\rangle\langle 00\cdots0| \otimes \rho)U^\dagger$

$$
= \left[\begin{array}{c:c} A_1 & \\ A_2 & \\ \vdots & W \\ A_{2^k} & \end{array}\right]
\begin{bmatrix} \rho & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}
\left[\begin{array}{cccc} A_1^\dagger & A_1^\dagger & \cdots & A_{2^k}^\dagger \\ \hdashline \multicolumn{4}{c}{W^\dagger} \end{array}\right]
$$

$$
= \begin{bmatrix} A_1\rho & 0 & \cdots & 0 \\ A_2\rho & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{2^k}\rho & 0 & \cdots & 0 \end{bmatrix}
\left[\begin{array}{cccc} A_1^\dagger & A_1^\dagger & \cdots & A_{2^k}^\dagger \\ \hdashline \multicolumn{4}{c}{W^\dagger} \end{array}\right]
$$

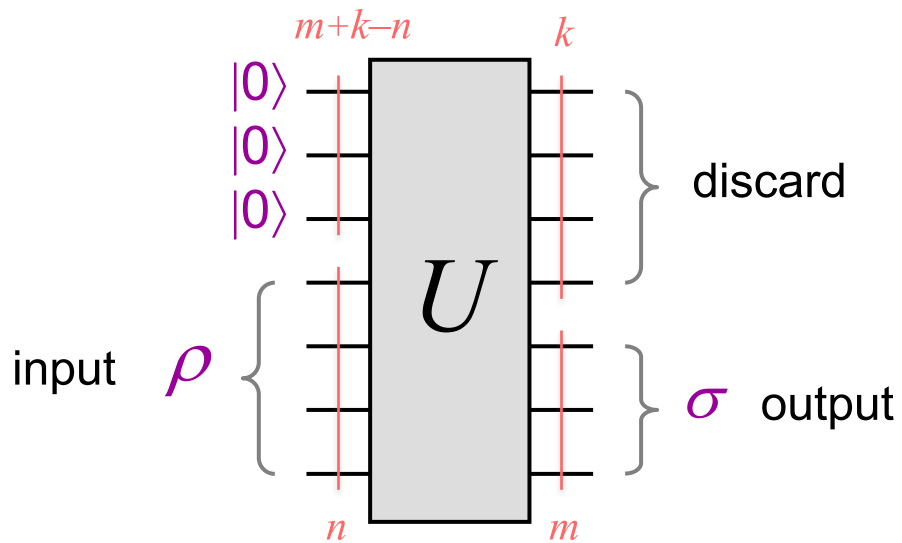
$$
= \begin{bmatrix} A_1\rho A_1^\dagger & A_1\rho A_2^\dagger & \cdots & A_1\rho A_{2^k}^\dagger \\ A_2\rho A_1^\dagger & A_2\rho A_2^\dagger & \cdots & A_2\rho A_{2^k}^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ A_{2^k}\rho A_1^\dagger & A_{2^k}\rho A_2^\dagger & \cdots & A_{2^k}\rho A_{2^k}^\dagger \end{bmatrix}
$$

Simulations among operations (5)

Tracing out the high-order k qubits of this state yields

$$A_1 \rho A_1^\dagger + A_2 \rho A_2^\dagger + \cdots + A_{2^k} \rho A_{2^k}^\dagger$$

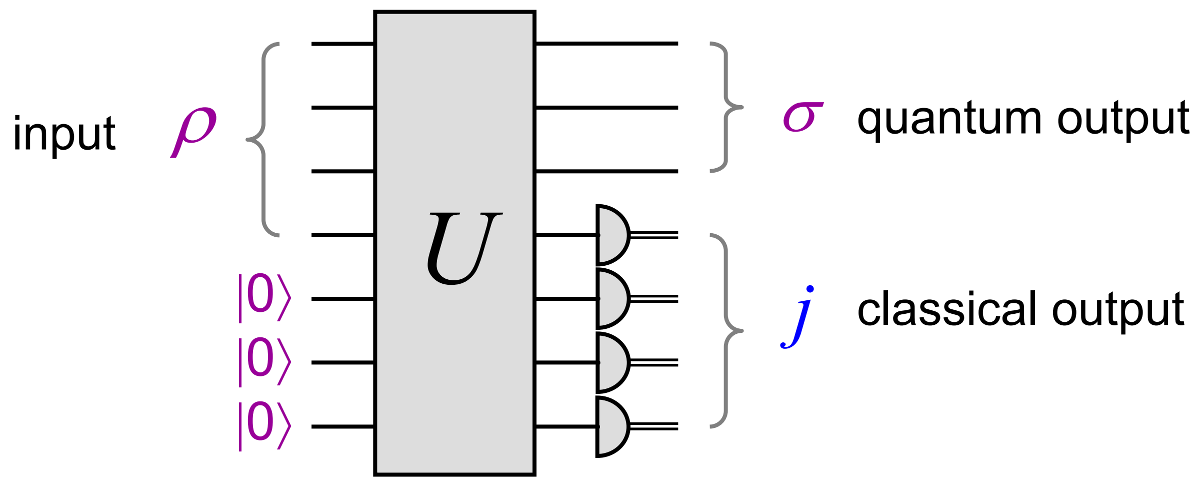
exactly the output of mapping that we want to simulate



Note: this approach is ***not always optimal*** in the number of ancilliary qubits used—there are more efficient methods

Simulations among operations (6)

Theorem 2: any ***POVM measurement*** can also be simulated by applying a unitary operation on a larger quantum system and then measuring:



This is the same diagram as for Theorem 1 (drawn with the extra qubits at the bottom) but where the “discarded” qubits are measured and part of the output