Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lectures 10–11 (2019)

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More state distinguishing problems

More state distinguishing problems

Which of these states are distinguishable? Divide them into equivalence classes:

1. $|0\rangle + |1\rangle$

2. $-|0\rangle - |1\rangle$

3. $\begin{cases} |0\rangle \text{ with prob. } ½ \\ |1\rangle \text{ with prob. } ½ \end{cases}$

4. $\begin{cases} |0\rangle + |1\rangle & \text{with prob. } ½ \\ |0\rangle - |1\rangle & \text{with prob. } ½ \end{cases}$

5. $\begin{cases} |0\rangle & \text{with prob. } ½ \\ |0\rangle + |1\rangle & \text{with prob. } ½ \end{cases}$

6. $\begin{cases} |0\rangle & \text{with prob. } ¼ \\ |1\rangle & \text{with prob. } ¼ \\ |0\rangle + |1\rangle & \text{with prob. } ¼ \\ |0\rangle - |1\rangle & \text{with prob. } ¼ \end{cases}$

7. The first qubit of $|01\rangle - |10\rangle$

Answers later on ...

This is a probabilistic mixed state (→ 3)

Density matrix formalism

Density matrices (1)

Until now, we've represented quantum states as ***vectors*** (e.g. $|\psi\rangle$, and all such states are called ***pure states***)

An alternative way of representing quantum states is in terms of ***density matrices*** (a.k.a. ***density operators***)

The density matrix of a pure state $|\psi\rangle$ is the matrix $\rho = |\psi\rangle\langle\psi|$

Example: the density matrix of $\alpha|0\rangle + \beta|1\rangle$ is

$$\rho = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}$$

Density matrices (2)

How do quantum operations work using density matrices?

Effect of a unitary operation on a density matrix:
applying U to ρ yields $U\rho U^\dagger$

(this is because the modified state is $U|\psi\rangle\langle\psi|U^\dagger$)

Effect of a measurement on a density matrix:
measuring state ρ with respect to the basis $|\varphi_1\rangle$, $|\varphi_2\rangle, \ldots,$ $|\varphi_d\rangle$, yields the k^{th} outcome with probability $\langle\varphi_k|\rho|\varphi_k\rangle$

(this is because $\langle\varphi_k|\rho|\varphi_k\rangle = \langle\varphi_k|\psi\rangle\langle\psi|\varphi_k\rangle = |\langle\varphi_k|\psi\rangle|^2$)

—and the state collapses to $|\varphi_k\rangle\langle\varphi_k|$

Density matrices (3)

A probability distribution on pure states is called a ***mixed state***:

$\big((|\psi_1\rangle, p_1), (|\psi_2\rangle, p_2), \ldots, (|\psi_d\rangle, p_d) \big)$

The ***density matrix*** associated with such a mixed state is:

$$\rho = \sum_{k=1}^{d} p_k |\psi_k\rangle\langle\psi_k|$$

Example: the density matrix for $\big((|0\rangle, \tfrac{1}{2}), (|1\rangle, \tfrac{1}{2}) \big)$ is:

$$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Question: what is the density matrix of

$\big((|0\rangle + |1\rangle, \tfrac{1}{2}), (|0\rangle - |1\rangle, \tfrac{1}{2}) \big)$?

Density matrices (4)

How do quantum operations work for these ***mixed*** states?

Effect of a unitary operation on a density matrix:

applying U to ρ ***still*** yields $U\rho U^\dagger$

This is because the modified state is:

$$\sum_{k=1}^{d} p_k U|\psi_k\rangle\langle\psi_k|U^\dagger = U\left(\sum_{k=1}^{d} p_k|\psi_k\rangle\langle\psi_k|\right)U^\dagger = U\rho U^\dagger$$

Effect of a measurement on a density matrix:

measuring state ρ with respect to the basis $|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_d\rangle$,

still yields the k^{th} outcome with probability $\langle\varphi_k|\rho|\varphi_k\rangle$

Why?

Recap: density matrices

Quantum operations in terms of density matrices:

- Applying U to ρ yields $U\rho U^\dagger$

- Measuring state ρ with respect to the basis $|\varphi_1\rangle, |\varphi_2\rangle, \dots, |\varphi_d\rangle$, yields: k^{th} outcome with probability $\langle\varphi_k|\rho|\varphi_k\rangle$
—and causes the state to collapse to $|\varphi_k\rangle\langle\varphi_k|$

Since these are expressible in terms of density matrices alone (independent of any specific probabilistic mixtures), states with identical density matrices are ***operationally indistinguishable***

Return to state distinguishing problems …

State distinguishing problems (1)

The ***density matrix*** of the mixed state

$((|\psi_1\rangle, p_1), (|\psi_2\rangle, p_2), \ldots, (|\psi_d\rangle, p_d))$ is: $\displaystyle\sum_{k=1}^{d} p_k |\psi_k\rangle\langle\psi_k|$

Examples (from earlier in lecture):

1. & 2. $|0\rangle + |1\rangle$ and $-|0\rangle - |1\rangle$ both have $\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. $\begin{cases} |0\rangle \text{ with prob. } ½ \\ |1\rangle \text{ with prob. } ½ \end{cases}$

4. $\begin{cases} |0\rangle + |1\rangle \text{ with prob. } ½ \\ |0\rangle - |1\rangle \text{ with prob. } ½ \end{cases}$

6. $\begin{cases} |0\rangle \text{ with prob. } ¼ \\ |1\rangle \text{ with prob. } ¼ \\ |0\rangle + |1\rangle \text{ with prob. } ¼ \\ |0\rangle - |1\rangle \text{ with prob. } ¼ \end{cases}$

(3., 4. and 6. all have) $\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

State distinguishing problems (2)

Examples (continued):

5. $\begin{cases} |0\rangle & \text{with prob. } ½ \\ |0\rangle + |1\rangle & \text{with prob. } ½ \end{cases}$

has: $$\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 \\ 1/2 & 1/4 \end{bmatrix}$$

7. The first qubit of $|01\rangle - |10\rangle$ **...? (later)**

Characterizing density matrices

$$\rho = \sum_{k=1}^{d} p_k |\psi_k\rangle\langle\psi_k|$$

Three properties of ρ :

- Tr$\rho = 1$ (Tr$M = M_{11} + M_{22} + \ldots + M_{dd}$)
- $\rho = \rho^\dagger$ (i.e. ρ is *Hermitian)*
- $\langle\varphi|\rho|\varphi\rangle \geq 0$, for all states $|\varphi\rangle$ (i.e. ρ is *positive semidefinite*)

Moreover, for ***any*** matrix ρ satisfying the above properties, there exists a probabilistic mixture whose density matrix is ρ

Exercise: show this

Taxonomy of various normal matrices

Normal matrices

Definition: A matrix M is ***normal*** if $M^\dagger M = MM^\dagger$

Theorem: M is normal iff there exists a unitary U such that $M = U^\dagger D U$, where D is diagonal (i.e. unitarily diagonalizable)

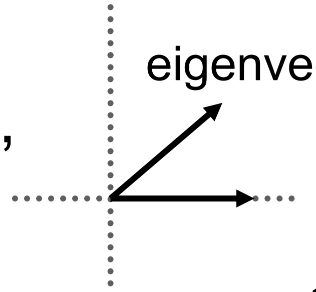
$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix}$$

Examples of ***ab***normal matrices:

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not even diagonalizable

$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable, but not unitarily

eigenvectors:



Unitary and Hermitian matrices

Normal: $$M = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix}$$ with respect to ***some*** orthonormal basis

Unitary: $M^\dagger M = I$ which implies $|\lambda_k|^2 = 1$, for all k

Hermitian: $M = M^\dagger$ which implies $\lambda_k \in \mathbb{R}$ for all k

Question: which matrices are both unitary ***and*** Hermitian?

Answer: reflections ($\lambda_k \in \{+1, -1\}$, for all k)

Positive semidefinite

Positive semidefinite: Hermitian and $\lambda_k \geq 0$, for all k

Theorem: M is positive semidefinite iff M is Hermitian and, for all $|\varphi\rangle$, $\langle\varphi|M|\varphi\rangle \geq 0$

(**Positive *definite***: $\lambda_k > 0$, for all k)

Projectors and density matrices

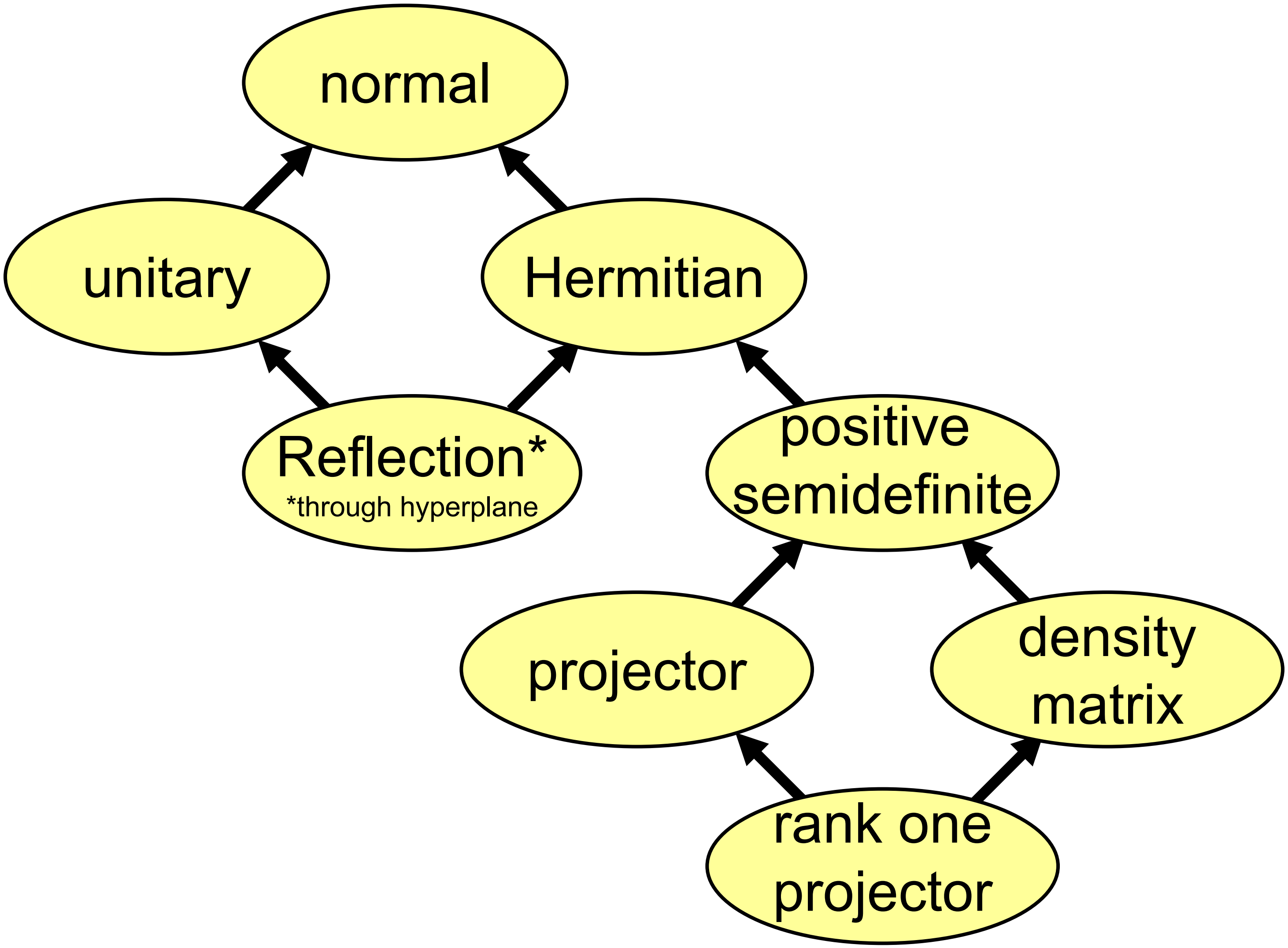
Projector: Hermitian and $M^2 = M$, which implies that M is positive semidefinite and $\lambda_k \in \{0,1\}$, for all k

Density matrix: positive semidefinite and $\mathrm{Tr}\, M = 1$, so $\sum_{k=1}^{d} \lambda_k = 1$

Question: which matrices are both projectors ***and*** density matrices?

Answer: rank-1 projectors ($\lambda_k = 1$ if $k = j$; otherwise $\lambda_k = 0$)

Taxonomy of normal matrices



Bloch sphere for qubits

Bloch sphere for qubits (1)

Consider the set of all 2x2 density matrices ρ

They have a nice representation in terms of the ***Pauli matrices***:

$$\sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \sigma_y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Note that these matrices—combined with I—form a ***basis*** for the vector space of all 2x2 matrices

We will express density matrices ρ in this basis

Note: coefficient of I must be ½, since X, Y, Z are traceless

Bloch sphere for qubits (2)

We will express $\rho = \dfrac{I + c_x X + c_y Y + c_z Z}{2}$

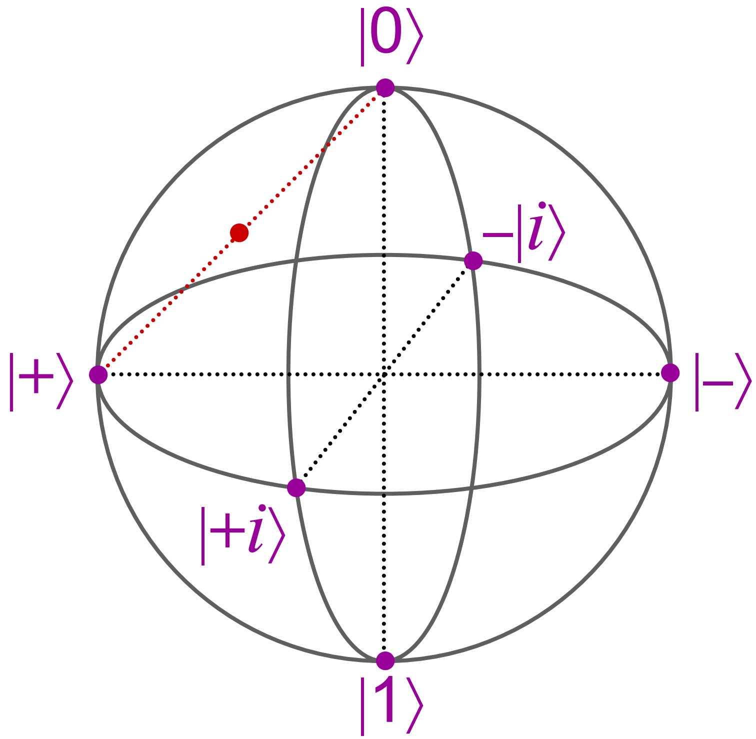
First consider the case of pure states $|\psi\rangle\langle\psi|$, where, without loss of generality, $|\psi\rangle = \cos(\theta)|0\rangle + e^{2i\phi}\sin(\theta)|1\rangle$ $(\theta, \phi \in [0,\pi])$

$$\rho = \begin{bmatrix} \cos^2\theta & e^{-i2\varphi}\cos\theta\sin\theta \\ e^{i2\varphi}\cos\theta\sin\theta & \sin^2\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\cos(2\theta) & e^{-i2\varphi}\sin(2\theta) \\ e^{i2\varphi}\sin(2\theta) & 1-\cos(2\theta) \end{bmatrix}$$

Therefore $c_z = \cos(2\theta)$, $c_x = \cos(2\phi)\sin(2\theta)$, $c_y = \sin(2\phi)\sin(2\theta)$

These are ***polar coordinates*** of a unit vector $(c_x, c_y, c_z) \in \mathbb{R}^3$

Bloch sphere for qubits (3)



$|+\rangle = |0\rangle + |1\rangle$

$|-\rangle = |0\rangle - |1\rangle$

$|+i\rangle = |0\rangle + i|1\rangle$

$|-i\rangle = |0\rangle - i|1\rangle$

Note that ***orthogonal*** corresponds to ***antipodal*** here

Pure states are on the surface, and mixed states are inside (being weighted averages of pure states)

Distinguishing mixed states

Distinguishing mixed states (1)

What's the best distinguishing strategy between these two mixed states?

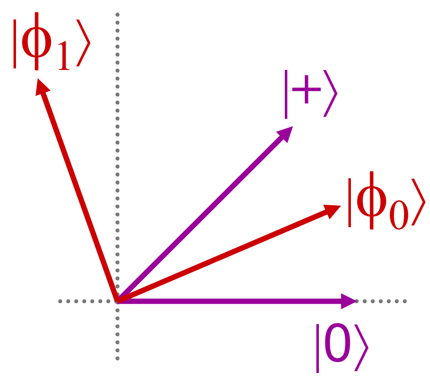
$$\begin{cases} |0\rangle & \text{with prob. } 1/2 \\ |0\rangle + |1\rangle & \text{with prob. } 1/2 \end{cases}$$

$$\rho_1 = \begin{bmatrix} 3/4 & 1/2 \\ 1/2 & 1/4 \end{bmatrix}$$

$$\begin{cases} |0\rangle & \text{with prob. } 1/2 \\ |1\rangle & \text{with prob. } 1/2 \end{cases}$$

$$\rho_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

ρ_1 also arises from this orthogonal mixture:



… as does ρ_2 from:

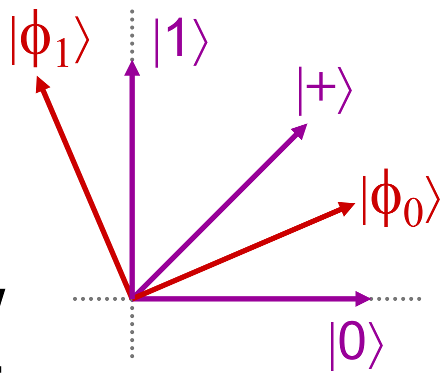
$$\begin{cases} |\phi_0\rangle & \text{with prob. } \cos^2(\pi/8) \\ |\phi_1\rangle & \text{with prob. } \sin^2(\pi/8) \end{cases}$$

$$\begin{cases} |\phi_0\rangle & \text{with prob. } 1/2 \\ |\phi_1\rangle & \text{with prob. } 1/2 \end{cases}$$

Distinguishing mixed states (2)

We've effectively found an orthonormal basis $|\phi_0\rangle$, $|\phi_1\rangle$ in which both density matrices are diagonal:

$$\rho_2' = \begin{bmatrix} \cos^2(\pi/8) & 0 \\ 0 & \sin^2(\pi/8) \end{bmatrix} \qquad \rho_1' = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Rotating $|\phi_0\rangle$, $|\phi_1\rangle$ to $|0\rangle$, $|1\rangle$ the scenario can now be examined using classical probability theory:

Distinguish between two ***classical*** coins, whose probabilities of "heads" are $\cos^2(\pi/8)$ and ½ respectively (details: exercise)

Question: what do we do if we aren't so lucky to get two density matrices that are simultaneously diagonalizable?

general quantum operations

more commonly known as

quantum channels

General quantum operations (1)

Also known as:
"quantum channels"
"completely positive trace preserving maps",
"admissible operations"

Let $A_1, A_2, \ldots, A_m$ be matrices satisfying $\sum_{j=1}^{m} A_j^\dagger A_j = I$

Then the mapping $\rho \mapsto \sum_{j=1}^{m} A_j \rho A_j^\dagger$ is a general quantum op

Note: $A_1, A_2, \ldots, A_m$ do not have to be square matrices

Example 1 (unitary op): applying U to ρ yields $U\rho U^\dagger$

General quantum operations (2)

Example 2 (decoherence): let $A_0 = |0\rangle\langle 0|$ and $A_1 = |1\rangle\langle 1|$

This quantum op maps ρ to $|0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$

For $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} \mapsto \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}$

Corresponds to measuring ρ "without looking at the outcome"

After looking at the outcome, ρ becomes $\begin{cases} |0\rangle\langle 0| & \text{with prob. } |\alpha|^2 \\ |1\rangle\langle 1| & \text{with prob. } |\beta|^2 \end{cases}$

General quantum operations (3)

Example 3

Let $A_0 = I \otimes \langle 0| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and $A_1 = I \otimes \langle 1| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- Any state of the form $\rho \otimes \sigma$ (product state) becomes ρ
- State $\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right)\left(\frac{1}{\sqrt{2}}\langle 00| + \frac{1}{\sqrt{2}}\langle 11|\right)$ becomes $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

  It's the same density matrix as for $((\frac{1}{2}, |0\rangle), (\frac{1}{2}, |1\rangle))$

- Corresponds to "discarding the second register"

The operation is called the ***partial trace*** $\mathrm{Tr}_2\, \rho$