Assignment 2

Due: 11:59pm, October 2, 2026

1. **Distinguishing between pairs of unitaries [12 points].**

   In each case, you are given a black-box gate that computes one of the two given unitaries, but you are not told which one. Your goal is to determine which of the two unitaries it is. To do this, you can create any quantum state, apply the black-box gate to this state, and then measure in some basis (that is, you can apply a unitary of your choosing and then measure in the computational basis). You can only use the black-box gate once.

   For example, consider the case where the two unitaries are $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In this case, setting the initial state to $|+\rangle$, applying the black-box unitary, followed by H and measuring yields: 0 in the first case and 1 in the second case. So this is a perfect distinguishing procedure (it succeeds with probability 1).

   Give a perfect distinguishing procedure in each case below.

   (a) $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

   (b) $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

   (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

2. **Distinguishing between 2-qubit states with access to only one qubit [6 points].**

   Suppose that you want to distinguish between these two 2-qubit states

$$|\phi_0\rangle = |+\rangle|+\rangle \tag{1}$$
$$|\phi_1\rangle = \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle, \tag{2}$$

   but you only have access to the first of the two qubits. (You might imagine that the first qubit is in your lab and the second qubit is in someone else's lab, outside your control.)

   So you can apply a measurement of your choosing to the first qubit and, on the basis of the outcome, guess which of the two states it was.

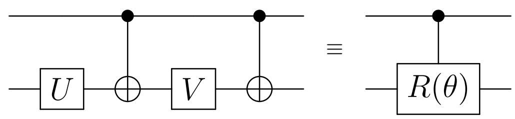
   (a) Give a procedure that succeeds with probability $\begin{cases} 1 & \text{if the state is } |\phi_0\rangle \\ \frac{1}{2} & \text{if the state is } |\phi_1\rangle. \end{cases}$

   (b) Give a procedure that succeeds with probability $\frac{2}{3}$ in both of the cases.
   (Hint: try randomly modifying the outcome of the procedure in part (a).)

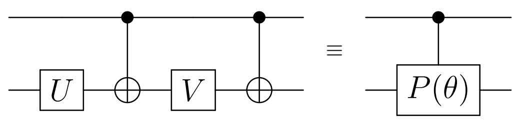
3. **Simulating controlled-U gates with CNOT and 1-qubit gates [12 points].**

   Let $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $P(\theta) = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$, where $\theta \in [0, 2\pi)$.

   (a) Show that, for all $\theta \in [0, 2\pi)$, there exist 1-qubit unitaries U and V such that

   (b) Show that, for all $\theta \in [0, 2\pi)$, there exist 1-qubit unitaries U and V such that

   (c) For all $\theta_1, \theta_2 \in [0, 2\pi)$, show how to simulate a controlled-$\big(R(\theta_1)P(\theta_2)\big)$ gate with two CNOT gates and up to three 1-qubit unitaries.

4. **State distinguishing problems involving mixed states [12 points].**

   In each case, one of the two given states is prepared and sent to you (you are not told which one). Your goal is to guess which of the two states it is.

   Describe a distinguishing procedure and give its worst-case success probability. Your grade will depend on how close your worst-case success probability is to optimal.

   (a) $|+\rangle$ and $\begin{cases} |0\rangle & \text{with probability } \frac{1}{2} \\ |1\rangle & \text{with probability } \frac{1}{2} \end{cases}$

   (b) $|0\rangle$ and $\begin{cases} |0\rangle & \text{with probability } \frac{1}{2} \\ |1\rangle & \text{with probability } \frac{1}{2} \end{cases}$

   (c) $\begin{cases} |0\rangle & \text{with probability } \frac{1}{2} \\ i\,|1\rangle & \text{with probability } \frac{1}{2} \end{cases}$ and $\begin{cases} |0\rangle & \text{with probability } \frac{1}{2} \\ -i\,|1\rangle & \text{with probability } \frac{1}{2} \end{cases}$

5. **Basic questions about density matrices [6 points].**

   (a) Calculate the density matrix of this probabilistic mixture of the three trine states:

   $$\begin{cases} |0\rangle & \text{with probability } \frac{1}{3} \\ -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle & \text{with probability } \frac{1}{3} \\ -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle & \text{with probability } \frac{1}{3}. \end{cases}$$

   (b) For a square matrix M, let M^T denote the *transpose* of M (that is, $(M^T)_{ij} = M_{ji}$). Either give an example of a 2×2 density matrix ρ such that $\rho^T \neq \rho$ or show that no such state exists.

(c) Give a probabilistic mixture of *orthonormal* states with the same density matrix as

$$\begin{cases} |0\rangle & \text{with probability } \frac{1}{2} \\ |+\rangle & \text{with probability } \frac{1}{2}. \end{cases}$$

6. **(This is an optional challenge question)**
State distinguishing problems involving mixed states [0 points].

Show that every 2×2 density matrix ρ can be expressed as an *equally weighted mixture* of pure states. That is

$$\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$$

for states $|\psi_1\rangle$ and $|\psi_2\rangle$ (note that, in general, the two states will not be orthogonal).

(Hint: one approach is to think geometrically about the positions of the states ρ, $|\psi_1\rangle\langle\psi_1|$, and $|\psi_2\rangle\langle\psi_2|$ on the Bloch sphere.)