Assignment 6

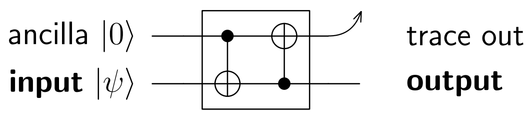
Due: 11:59pm, November 25, 2025

1. **Some Stinespring channels and their Kraus forms [18 points; 6 each].**

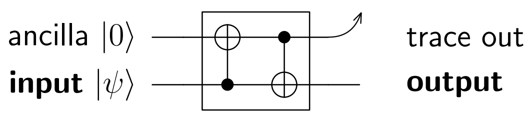
   In each case below, a quantum channel that maps 1-qubit to 1-qubit is described in the Stinespring form.

   In each case:
   (i) Describe how the channel affects an input state of the form $\alpha_0|0\rangle + \alpha_1|1\rangle$. Give the output state as a 2×2 density matrix (whose entries are functions of α_0 and α_1).
   (ii) Give Kraus operators for the channel.

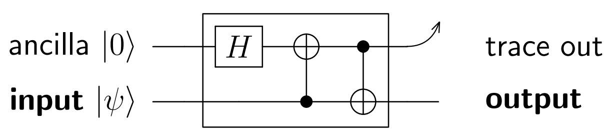
   (a) The channel



   (b) The channel



   (c) The channel



2. **An application of the Holevo-Helstrom Theorem [12 points].**

   Using the Holevo-Helstrom Theorem, prove that the best average-case success probability for distinguishing between $|0\rangle$ and $|+\rangle$ is not higher than $\cos^2(\pi/8) = \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$.

3. **Quantum error-correcting code for the one-out-of-two-X-error noisy channel [15 points].**

   Consider the *one-out-of-two-X-error* channel, which applies the following noise model to two qubits:

$$\begin{cases} X \otimes I & \text{with probability } \frac{1}{2} \\ I \otimes X & \text{with probability } \frac{1}{2}. \end{cases}$$

   Suppose that you want to communicate a qubit through this noisy channel.

   Describe:

   - an *encoding* procedure that maps one qubit of data to a 2-qubit encoding, and
   - a *decoding* procedure that maps a 2-qubit state to a 1-qubit state,

   such that, if the encoding a qubit of data is subjected to the one-out-of-two-X-error channel, then the decoding procedure recovers the original data.

4. **Quantum error-correcting code for the even-number-of-X-out-of-three noisy channel [15 points].**

   Consider the *even-number-of-X-out-of-three* channel that takes three qubits as input and acts on them as

$$\begin{cases} I \otimes I \otimes I & \text{with probability } \frac{1}{4} \\ X \otimes X \otimes I & \text{with probability } \frac{1}{4} \\ X \otimes I \otimes X & \text{with probability } \frac{1}{4} \\ I \otimes X \otimes X & \text{with probability } \frac{1}{4}. \end{cases}$$

   Suppose that Alice wants to send Bob a qubit through this channel, without incurring any error.

   Describe:

   - an *encoding* procedure that maps one qubit of data to a 3-qubit encoding, and
   - a *decoding* procedure that maps a 3-qubit state to a 1-qubit state,

   such that, if the encoding a qubit of data is subjected to the even-number-of-X-out-of-three channel, then the decoding procedure recovers the original data.

5. **(This is an optional question for bonus credit)**
   Stinespring form of a channel with particular properties [8 points].

   Consider a channel that has the following effect on the four computational basis states:

input	output
$\lvert 00\rangle$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\lvert 00\rangle$
$\lvert 10\rangle$	$\lvert 10\rangle$
$\lvert 11\rangle$	$\lvert 11\rangle$

   Notice that this is not unitary, since it maps both $\lvert 00\rangle$ and $\lvert 01\rangle$ to the state $\lvert 00\rangle$.

   Describe a Stinespring form of a channel with this property *that uses only one ancilla qubit*. (So it adds *one* ancilla qubit, performs a unitary operation on the 3-qubit system, and then traces out the ancilla.)

   There is a fairly simple solution that can be explained in less than one page. If you submit a solution to this question then please explain it clearly and do not exceed two pages.