Assignment 5

Due: 11:59pm, November 13, 2025

1. **Partial trace [15 points; 5 each].**

   For the 3-qubit state $|\psi\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|110\rangle + \frac{1}{2}|001\rangle - \frac{1}{2}|111\rangle$, answer the following:

   (a) Give the 4×4 density matrix of the state that results if the third qubit is traced out.

   (b) Prove that the 4×4 density matrix of the state of the first two qubits (i.e., the state that results when the third qubit traced out) is *separable* by expressing it as

   $$p_0\, \rho_0 \otimes \sigma_0 \ + \ p_1\, \rho_1 \otimes \sigma_1,$$

   where ρ_0, ρ_1, σ_0, σ_1 are 2×2 density matrices and p_0, $p_1 \geq 0$.

   (c) Suppose that Alice possesses the first qubit and Bob possesses *the second and the third* qubit of state $|\psi\rangle$ and they want to construct a Bell state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. Describe local unitaries that Alice and Bob can perform on their respective systems (where Alice's system is the first qubit, and Bob's system is the second and third qubit) that cause the state of the first two qubits to become $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

2. **Effect of 1-qubit quantum channel on 2-qubit system [15 points; 5 each].**

   Suppose that a 2-qubit system is in state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. For each channel below, give the 4×4 density matrix of the state of the 2-qubit system after the channel has been applied to the second qubit (the density matrix will be a function of p).

   (a) The phase damping channel with parameter $p \in [0, \frac{1}{2}]$.

   (b) The depolarizing channel with parameter $p \in [0, \frac{3}{4}]$.

   (c) The amplitude damping channel with parameter $p \in [0, 1]$.

3. **Quantum channels with certain properties? [15 points; 3 each].**

   In each case below, either describe a quantum channel using Kraus operators that has the given property or state that "no such channel exists" (justification is not necessary if no such channel exists).

   (a) The property that it maps any density matrix of the form $\begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}$ to $\begin{bmatrix} \rho_{00} & \frac{1}{2}\rho_{01} \\ \frac{1}{2}\rho_{10} & \rho_{11} \end{bmatrix}$.

   (b) The property that it maps $|0\rangle\langle 0|$ to $|0\rangle\langle 0|$, and $|1\rangle\langle 1|$ to $|+\rangle\langle +|$.

   (c) The property that it maps $|0\rangle\langle 0|$ to $|0\rangle\langle 0|$, and $|+\rangle\langle +|$ to $|1\rangle\langle 1|$.

   (d) The property that it maps $|00\rangle\langle 00|$ to $|0\rangle\langle 0|$, $|01\rangle\langle 01|$ to $|0\rangle\langle 0|$, $|10\rangle\langle 10|$ to $|0\rangle\langle 0|$, and $|11\rangle\langle 11|$ to $|1\rangle\langle 1|$ (in the computational basis, this is like an "and" gate).

   (e) The property that it maps $|0\rangle\langle 0|$ to $|0\rangle\langle 0|$, $|1\rangle\langle 1|$ to $|1\rangle\langle 1|$, and $|+\rangle\langle +|$ to $|-\rangle\langle -|$.

4. **Probabilistic mixtures of Bell states [15 points; 5 each].**
Consider this probabilistic mixture of Bell states

$$\begin{cases} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle & \text{with probability } \frac{1}{3} + \delta \\ \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle & \text{with probability } \frac{1}{3} - \delta \\ \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle & \text{with probability } \frac{1}{6} \\ \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle & \text{with probability } \frac{1}{6}, \end{cases}$$

where δ is a parameter such that $0 \le \delta \le \frac{1}{3}$.

(a) Give the 4×4 density matrix for this state. Write out the entries of this matrix.

(b) Give the density matrix for the special case where $\delta = 0$ (write out its entries). Is that state separable or entangled?

Give the density matrix for the special case where $\delta = \frac{1}{3}$ (write out its entries). Is that state separable or entangled?

(c) For what values of $\delta \in [0, \frac{1}{3}]$ is the state separable and for what values is it entangled? Justify your answer.

Hint: You may use the positive partial transpose test to determine whether or not the state is separable.

5. **(This is an optional question for bonus credit)**
State conversion with local operations and classical communication [8 points].

Suppose that Alice and Bob each possess a qutrit in their respective labs and their joint state is

$$|\psi\rangle = \tfrac{1}{\sqrt{3}}|00\rangle + \tfrac{1}{\sqrt{3}}|11\rangle + \tfrac{1}{\sqrt{3}}|22\rangle.$$

Suppose that they want to transform this state to $|\phi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$, using:

- local operations (which can be local unitaries and local measurements),
- *classical* communication

but they are forbidden from having any *quantum* communication.

Explain how they can transform $|\psi\rangle$ to $|\phi\rangle$ in this manner, where the classical communication is *one classical message from Alice to Bob.*

There is a solution that can be explained in less than one page. If you submit a solution to this question then please explain it clearly and do not exceed two pages.