Assignment 4
Due: 11:59pm, November 3, 2025

1. **Grover's algorithm for some densities of satisfying inputs [15 points; 5 each].**

   Let $f : \{0,1\}^n \to \{0,1\}$ (where $n \geq 2$). Recall that Grover's algorithm creates the initial state $H|00...0\rangle|-\rangle$ and then iterates the operation $-HU_0HU_f$.

   In each case below, determine the state after *one single iteration* of Grover's algorithm. Also, what's the probability that, if this state is measured, the outcome is a satisfying input to f?

   (a) The case where f has no satisfying inputs.

   (b) The case where f has $\frac{1}{4}2^n$ satisfying inputs.

   (c) The case where f has $\frac{3}{4}2^n$ satisfying inputs.

2. **A version of Grover's algorithm for domains of arbitrary size [15 points].**

   The version of Grover's algorithm that we have seen in the course assumes that the domain is a set of the form $\{0,1\}^n$. Here we describe the algorithm for domains of arbitrary size $m \geq 4$. The goal is to find a satisfying input to $f : \{1,2,\ldots,m\} \to \{0,1\}$, which is given as a black box. For simplicity, assume that f has a *unique* satisfying input, which we refer to as r (so we're assuming that $f(x) = 0$ for all $x \neq r$; and $f(r) = 1$).

   Define the $m \times m$ matrix

   $$M = \begin{bmatrix} 1 - \frac{2}{m} & -\frac{2}{m} & \cdots & -\frac{2}{m} \\ -\frac{2}{m} & 1 - \frac{2}{m} & \cdots & -\frac{2}{m} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{2}{m} & -\frac{2}{m} & \cdots & 1 - \frac{2}{m} \end{bmatrix}. \tag{1}$$

   (a) [10 points] Prove that M is unitary and give the eigenvalues and eigenvectors of M. (Hint: what is $M(|1\rangle + |2\rangle + \cdots + |m\rangle)$?)

   (b) [5 points] Consider the 2-dimensional space spanned by

   $$|A_0\rangle = \frac{1}{\sqrt{m-1}}\sum_{x \neq r}|x\rangle \quad \text{and} \quad |A_1\rangle = |r\rangle, \tag{2}$$

   and define U_f as the $m \times m$ unitary operation such that

   $$U_f|x\rangle = (-1)^{f(x)}|x\rangle \tag{3}$$

   for each $x \in \{1,2,\ldots,m\}$. Prove that, in the 2-dimensional space spanned by $|A_0\rangle$ and $|A_1\rangle$, the effect of $-MU_f$ is a rotation by angle 2θ, where $\sin(\theta) = \frac{1}{\sqrt{m}}$.

   (The search algorithm starts with the state $\frac{1}{\sqrt{m}}(|1\rangle + |2\rangle + \cdots + |m\rangle)$ and then applies MU_f approximately $\frac{\pi}{4}\sqrt{m}$ times, to get a state close to $|r\rangle$.)

3. **State distinguishing problems involving mixed states [15 points; 5 each].**

   In each case, one of the two given states is prepared and sent to you (you are not told which one). Your goal is to guess which of the two states it is.

   Describe a distinguishing procedure and give its worst-case success probability. Your grade will depend on how close your worst-case success probability is to optimal.

   (Hint: for some of these, consider a methodology like the one used in question 2 on Assignment 2.)

   (a) $|+\rangle$ and $\begin{cases} |0\rangle & \text{with probability } \frac{1}{2} \\ |1\rangle & \text{with probability } \frac{1}{2} \end{cases}$

   (b) $|0\rangle$ and $\begin{cases} |0\rangle & \text{with probability } \frac{1}{2} \\ |1\rangle & \text{with probability } \frac{1}{2} \end{cases}$

   (c) $\begin{cases} |0\rangle & \text{with probability } \frac{1}{2} \\ i|1\rangle & \text{with probability } \frac{1}{2} \end{cases}$ and $\begin{cases} |0\rangle & \text{with probability } \frac{1}{2} \\ -i|1\rangle & \text{with probability } \frac{1}{2} \end{cases}$

4. **Basic questions about density matrices [15 points; 3 each].**

   (a) Calculate the density matrix of this probabilistic mixture of the three trine states:

   $$\begin{cases} |0\rangle & \text{with probability } \frac{1}{3} \\ -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle & \text{with probability } \frac{1}{3} \\ -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle & \text{with probability } \frac{1}{3}. \end{cases}$$

   (b) Either give an example of a 2×2 density matrix ρ such that $\rho + X\rho X \neq I$ or show that no such state exists. (As usual, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.)

   (c) For a square matrix M, let M^T denote the *transpose* of M (that is, $(M^T)_{ij} = M_{ji}$). Either give an example of a 2×2 density matrix ρ such that $\rho^T \neq \rho$ or show that no such state exists.

   (d) Give a probabilistic mixture of *orthonormal* states with the same density matrix as

   $$\begin{cases} |0\rangle & \text{with probability } \frac{1}{2} \\ |+\rangle & \text{with probability } \frac{1}{2}. \end{cases}$$

   (e) Show that every 2×2 density matrix ρ can be expressed as an *equally weighted mixture* of pure states. That is

   $$\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$$

   for states $|\psi_1\rangle$ and $|\psi_2\rangle$ (note that, in general, the two states will not be orthogonal). (Hint: one approach is to think geometrically about the positions of the states ρ, $|\psi_1\rangle\langle\psi_1|$, and $|\psi_2\rangle\langle\psi_2|$ on the Bloch sphere.)

5. (**This is an optional question for bonus credit**)
   Search problem for $\frac{3}{4}$ density of satisfying inputs [8 points].

   Let $f : \{0,1\}^n \to \{0,1\}$ (where $n \geq 2$) and assume that f has $\frac{3}{4}2^n$ satisfying inputs. Does there exist a quantum algorithm that makes just one f-query and is guaranteed to find a satisfying input? Answer yes or no and justify your answer by: giving such an algorithm (if the answer is yes); or proving that no such algorithm exists (if the answer is no).

   (As usual, an f-query is the unitary operation that maps $|x\rangle|y\rangle$ to $|x\rangle|y \oplus f(x)\rangle$, for all $x \in \{0,1\}^n$ and $y \in \{0,1\}$.)

   There is a solution that can be explained in less than one page. If you submit a solution to this question then please explain it clearly and do not exceed two pages.