Assignment 2

Due: 11:59pm, September 29, 2025

1. **Distinguishing between pairs of unitaries [15 points; 5 each].**

   In each case, you are given a black-box gate that computes one of the two given unitaries, but you are not told which one. Your goal is to determine which of the two unitaries it is. To do this, you can create any quantum state, apply the black-box gate to this state, and then measure in some basis (that is, you can apply a unitary of your choosing and then measure in the computational basis). You can only use the black-box gate once.

   For example, consider the case where the two unitaries are $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In this case, setting the initial state to $|+\rangle$, applying the black-box unitary, followed by H and measuring yields: 0 in the first case and 1 in the second case. So this is a perfect distinguishing procedure (it succeeds with probability 1).

   Give a perfect distinguishing procedure in each case below.

   (a) $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

   (b) $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

   (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

2. **Distinguishing between 2-qubit states with access to only one qubit [15 points].**

   Suppose that you want to distinguish between these two 2-qubit states

$$|\phi_0\rangle = |+\rangle|+\rangle \tag{1}$$

$$|\phi_1\rangle = \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle, \tag{2}$$

   but you only have access to the first of the two qubits. (You might imagine that the first qubit is in your lab and the second qubit is in someone else's lab, outside your control.)

   So you can apply a measurement of your choosing to the first qubit and, on the basis of the outcome, guess which of the two states it was.

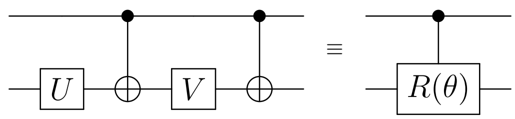
   (a) [10 points] Give a procedure that succeeds with probability $\begin{cases} 1 & \text{if the state is } |\phi_0\rangle \\ \frac{1}{2} & \text{if the state is } |\phi_1\rangle. \end{cases}$

   (b) [5] Give a procedure that succeeds with probability $\frac{2}{3}$ in both of the cases. (Hint: try randomly modifying the outcome of the procedure in part (a).)

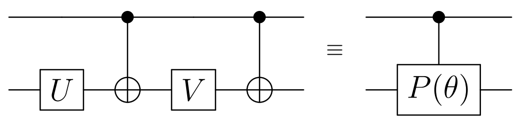
3. **Simulating controlled-U gates with CNOT and 1-qubit gates [15 points; 5 each].**

   Let $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $P(\theta) = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}$, where $\theta \in [0, 2\pi)$.

   (a) Show that, for all $\theta \in [0, 2\pi)$, there exist 1-qubit unitaries U and V such that

   (b) Show that, for all $\theta \in [0, 2\pi)$, there exist 1-qubit unitaries U and V such that

   (c) For all $\theta_1, \theta_2 \in [0, 2\pi)$, show how to simulate a controlled-$\big(R(\theta_1)P(\theta_2)\big)$ gate with two CNOT gates and up to three 1-qubit unitaries.

4. **Some simple black-box query problems [15 points].**

   In each part below, the goal is to identify the function, assuming it is one of the four given functions. Give a quantum algorithm that identifies the function with one single f-query, along with an explanation of why your algorithm works.

   (a) [10 points] The four functions are $f_{00}, f_{01}, f_{10}, f_{11} : \{0,1\}^2 \to \{0,1\}$ where, for each $a_1a_2 \in \{0,1\}^2$,

   $$f_{a_1a_2}(x_1, x_2) = (a_1x_1 + a_2x_2) \bmod 2. \tag{3}$$

   (b) [5] The functions are $f_{00}, f_{01}, f_{10}, f_{11} : \{0,1\}^3 \to \{0,1\}$ where, for each $a_1a_2 \in \{0,1\}^2$,

   $$f_{a_1a_2}(x_1, x_2, x_3) = (a_1x_1x_2 + a_2x_3 + a_1a_2) \bmod 2. \tag{4}$$

5. **(This is an optional question for bonus credit)**
   Fully identifying a function $f : \{0,1\} \to \{0,1\}$ [8 points].

   Recall that, in Deutsch's problem, we are given a black-box for an arbitrary function $f : \{0,1\} \to \{0,1\}$, but we are not required to fully identify which of the four possible functions f is. Here we consider the problem where the goal is to correctly guess which of the four functions f is.

   It's easy to deduce that, with a single *classical* f-query, the best worst-case success probability achievable is $\frac{1}{2}$.

   Give a *quantum* algorithm that makes a single f-query and correctly guesses f with worst-case success probability $\frac{3}{4}$. (To be clear, by *worst-case success probability*, we mean that your algorithm should succeed with probability at least $\frac{3}{4}$ for each of the four functions.) Include an explanation of why your algorithm works.

   There is a solution that can be explained within one page. If you submit a solution to this question then please do not exceed two pages.