Sample Assignment A

1. **Distinguishing between pairs of bit states.** In the first class, we discussed a game where a bit is set in one of two ways:

   set-up I: The bit is set to state 0.

   set-up II: The bit is randomly set to $\begin{cases} 0 & \text{with probability } \frac{1}{2} \\ 1 & \text{with probability } \frac{1}{2}. \end{cases}$

   You are then given the bit (and you know that it was set according to one of the two procedures above, but are not told which one). Your goal is to guess which set-up was used (either I or II).

   A sensible strategy that was considered in class was:

   Strategy A
   if the bit received is 0 then guess I
   if the bit received is 1 then guess II

   The *average-case* success probability of strategy A (assuming that set-up I and set-up II are chosen with uniform probability, i.e., each occurs with probability $\frac{1}{2}$) was calculated to be $\frac{3}{4}$; whereas the *worst-case* success probability of strategy A is $\frac{1}{2}$.

   (a) Prove that the average-case success probability achieved by *any* strategy is at most $\frac{3}{4}$ (assuming uniform probability for the choice of the set-up).

   (b) What is the highest possible worst-case success probability? Justify your answer.

2. **Distinguishing between pairs of qubit states.** Suppose one of these qubit states is prepared and sent to you (you are not told which one):

   $$|0\rangle \quad \text{vs.} \quad |+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle. \tag{1}$$

   Your goal is to guess which of the two states it is. You are allowed to perform any measurement operation on the state to help you with this goal. Technically, you are allowed to perform any unitary operation on the qubit and then measure with respect to the computational basis.

   Show that there is a distinguishing procedure with average-case success probability is $\cos^2(\pi/8) = 0.853...$ and prove that this is optimal over all strategies where the unitary operation preceding the measurement is a rotation by some angle $\theta \in [0, \pi/4]$.

   Note: Using techniques developed later in the course, we'll see that $\cos^2(\pi/8)$ is optimal for *any* measurement; the restriction to rotations is only to simplify the calculations in a direct proof.

3. **Proving that $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is not a product state.**

   Prove that $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ cannot be expressed as a product of two 1-qubit state vectors.