Assignment 5

Due: 11:59pm, November 19, 2024

1. **Applications of the Holevo-Helstrom Theorem [15 points].**
The *state distinguishing problem* is where you receive one of two quantum states, $|\psi_0\rangle$ or $|\psi_1\rangle$, occurring with probability $\frac{1}{2}$ each, and your goal is to guess which state it is.

   (a) [6 points] We have previously seen that, when the possible states are $|0\rangle$ and $|+\rangle$, there is a method for distinguishing with success probability $\cos^2(\frac{\pi}{8}) = \frac{1}{2} + \frac{\sqrt{2}}{4}$. Use the Holevo-Helstrom theorem to show that this is the *optimal* success probability for this problem.

   (b) [9 points] Let $\theta \in [0, \frac{\pi}{2}]$ be a given (and known) parameter and consider the state distinguishing problem where

$$|\psi_0\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$
$$|\psi_1\rangle = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle.$$

   Using the Holevo-Helstrom Theorem, give the optimal measurement procedure and it's success probability. Show your calculations.

2. **Calculation of a Schmidt decomposition [15 points].**
Consider the 2-qubit state

$$|\psi\rangle = \tfrac{7}{10}|00\rangle + \tfrac{1}{10}|01\rangle + \tfrac{1}{10}|10\rangle + \tfrac{7}{10}|11\rangle.$$

Find a Schmidt decomposition of this state, namely an orthonormal basis $|\phi_0\rangle, |\phi_1\rangle$ and an orthonormal basis $|\mu_0\rangle, |\mu_1\rangle$ and Schmidt coefficients $\alpha_0, \alpha_1 \geq 0$ such that

$$|\psi\rangle = \alpha_0|\phi_0\rangle|\mu_0\rangle + \alpha_1|\phi_1\rangle|\mu_1\rangle.$$

Show how you calculated these states and coefficients.

3. **Quantum error-correcting code for a peculiar noise model [15 points].**
Consider the *one-out-of-two-X-error* channel, which applies the following noise model to two qubits:

$$\begin{cases} X \otimes I & \text{with probability } \frac{1}{2} \\ I \otimes X & \text{with probability } \frac{1}{2}. \end{cases}$$

Suppose that you want to communicate a qubit through this noisy channel. Describe:

- an *encoding* procedure that maps one qubit of data to a 2-qubit encoding, and
- a *decoding* procedure that maps a 2-qubit state to a 1-qubit state,

such that, if the encoding a qubit of data is subjected to the one-out-of-two-X-error channel, then the decoding procedure recovers the original data.

4. **Mixed unitary channels [15 points].**
A channel is *mixed-unitary* if it can be expressed in terms of Kraus operators $A_0, \cdots, A_{m-1}$ that are of the form $A_k = \sqrt{p_k}\,U_k$, where $U_0, \dots, U_{m-1}$ are *unitary* and $(p_0, \dots, p_{m-1})$ is a probability vector. Such a channel is equivalent to applying:

$$\begin{cases} U_0 & \text{with probability } p_0 \\ \vdots & \vdots \\ U_{m-1} & \text{with probability } p_{m-1}. \end{cases}$$

Consider the one-qubit channel that maps every 2×2 density matrix to the density matrix

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Show that this is a mixed-unitary channel that can be expressed in terms of *four* Kraus operators, of the form $A_0 = \sqrt{p_0}\,U_0$, $A_1 = \sqrt{p_1}\,U_1$, $A_2 = \sqrt{p_2}\,U_2$, $A_3 = \sqrt{p_3}\,U_3$.

5. **(This is an optional question for bonus credit)**
Mixed unitary channels [6 points].
Prove that the channel in question 4 cannot be expressed as a mixed-unitary channel with only *three* Kraus operators.

Note: If you submit a solution to this question then there is a size-limit of one page.