Assignment 4

Due date: 11:59pm, November 8, 2024

1. **Analysis of Grover's algorithm for some special densities of satisfying inputs [15 points; 5 each].** Let $f : \{0,1\}^n \to \{0,1\}$ (where $n \geq 2$). Recall that Grover's algorithm creates the initial state $H|00...0\rangle|-\rangle$ and then iterates the operation $-HU_0HU_f$.

   In each case below, determine the state after *one single iteration* of Grover's algorithm. Also, what's the probability that, if this state is measured, the outcome is a satisfying input to f?

   (a) The case where f has no satisfying inputs.

   (b) The case where f has $\frac{1}{4}2^n$ satisfying inputs.

   (c) The case where f has $\frac{1}{2}2^n$ satisfying inputs.

2. **Basic questions about density matrices [15 points; 5 each].**

   (a) Calculate the density matrix of this probabilistic mixture of the three trine states:

$$\begin{cases} |0\rangle & \text{with probability } \frac{1}{3} \\ -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle & \text{with probability } \frac{1}{3} \\ -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle & \text{with probability } \frac{1}{3}. \end{cases}$$

   (b) A valid density matrix ρ is called a *pure state* if and only if $\rho = |\psi\rangle\langle\psi|$, for some state vector $|\psi\rangle$. Prove that ρ is a pure state if and only if $\mathrm{Tr}(\rho^2) = 1$.

   (c) Show that every 2×2 density matrix ρ can be expressed as an *equally weighted mixture* of pure states. That is

$$\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$$

   for states $|\psi_1\rangle$ and $|\psi_2\rangle$ (note that, in general, the two states will not be orthogonal). (Hint: one approach is to think geometrically about the positions of the states ρ, $|\psi_1\rangle\langle\psi_1|$, and $|\psi_2\rangle\langle\psi_2|$ on the Bloch sphere.)

3. **Action of unitary operations on the Bloch sphere [15 points; 5 each].**
   For every 2×2 unitary matrix U, the effect of applying U on a qubit can be viewed as a rotation of the states in the Bloch sphere. For example, it can be shown that

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

   corresponds to rotation by angle 2θ along the axis specified by states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. In each case below, the axis of rotation as well as the angle of rotation (each of which can depend on the parameter θ):

   (a) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (b) $\begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$ (c) $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

4. **Simple questions concerning Kraus operators [15 points; 5 each].** For each pair of matrices, state whether or they are valid Kraus operators, and if they are valid Kraus operators, describe in simple terms (i) the *measurement* as well as (ii) the *channel* associated with them (all the cases of valid Kraus operators have conceptually very simple descriptions):

   (a) $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

   (b) $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

   (c) $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

5. **(This is an optional question for bonus credit)**
   Expressing a qutrit as an equally weighted mixture of pure states [6 points]. Let ρ be any 3×3 matrix that is the density matrix of the mixed state of a qutrit. Show that there exist three normalized vectors $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle \in \mathbb{C}^3$, representing pure states, such that

$$\rho = \frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{1}{3}|\psi_2\rangle\langle\psi_2| + \frac{1}{3}|\psi_3\rangle\langle\psi_3|.$$

   Note that $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ are not required to be orthogonal here.

   There is a solution that can be explained in less than one page. If you submit a solution to this question, please do not exceed two pages.