Assignment 2
Due date: 11:59pm, October 1, 2024

1. **Distinguishing between three qutrit states [15 points].** Suppose that one of these three qutrit states

$$\begin{aligned} |\phi_0\rangle &= \tfrac{1}{\sqrt{3}}|0\rangle + \tfrac{1}{\sqrt{3}}|1\rangle - \tfrac{1}{\sqrt{3}}|2\rangle \\ |\phi_1\rangle &= \tfrac{1}{\sqrt{3}}|0\rangle - \tfrac{1}{\sqrt{3}}|1\rangle + \tfrac{1}{\sqrt{3}}|2\rangle \\ |\phi_2\rangle &= -\tfrac{1}{\sqrt{3}}|0\rangle + \tfrac{1}{\sqrt{3}}|1\rangle + \tfrac{1}{\sqrt{3}}|2\rangle \end{aligned}$$

is prepared and sent to you (you are not told which one; each case arises with probability $\frac{1}{3}$). Describe a measurement procedure to guess the state.

Grading: 13 points if your procedure succeeds with probability at least 3/4.
15 points if your procedure succeeds with probability at least 9/10.

You are allowed to perform an exotic measurement (where the state is combined with an additional register in state $|0\rangle$ and then a measurement is performed with respect to an orthonormal basis in the higher dimensional system).

(Hint: try to find three orthogonal states that are close to these states.)

2. **Entanglement among three qubits [15 points; 5 each].** Suppose Alice, Bob and Carol each possess a qubit and the joint state of the three qubits is $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$.

   (a) Suppose that Carol leaves the scene, taking her qubit with her, and without communicating with either Alice or Bob. Are Alice and Bob's qubits in state $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$? Either find α_0, α_1 such that the three qubit state is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes (\alpha_0|0\rangle + \alpha_1|1\rangle)$ or show that no such α_0, α_1 exist.

   (b) Suppose again that Carol leaves the scene, taking her qubit with her, but she is allowed to send one classical bit to Alice. Carol wants to help Alice and Bob transform their state into the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (and without Alice and Bob having to send any messages between each other). The framework is as follows:

      i. Carol applies some unitary operation U to her qubit, and then measures the qubit, yielding the classical bit b.

      ii. Carol sends just the classical bit b to Alice.

      iii. Alice applies a unitary operation, depending on b, to her qubit. In other words, Alice has two unitary operations V_0 and V_1, and she applies V_b to her qubit.

   At the end of this procedure, the two-qubit state of state of Alice and Bob's qubits should be $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Explain how to make this procedure work.

   (c) Is it possible for Alice, Bob and Carol to each possess a qubit such that the joint state of the three qubits $|\phi\rangle$ has both of the following properties at the same time?

   Property 1: $|\phi\rangle = (\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle) \otimes (\alpha_0|0\rangle + \alpha_1|1\rangle)$, for some α_0, α_1.

   Property 2: $|\phi\rangle = (\beta_0|0\rangle + \beta_1|1\rangle) \otimes (\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle)$, for some β_0, β_1.

   Justify your answer.

3. **Teleporting a CNOT gate between two labs [15 points; 5 each].** Suppose that Alice has a qubit in her lab and Bob has a qubit in his lab and they wish to implement a CNOT gate between their respective qubits. The most natural way to accomplish this is for Alice to send her qubit over to Bob, who can then perform the CNOT gate in his lab and then Bob sends the qubit back to Alice. This works but requires *quantum communication* between Alice and Bob.

   Can they accomplish this with only *classical communication*? If they share two Bell states then the answer is yes. Using one Bell state, Alice can teleport her qubit over to Bob, who can the perform the CNOT gate, and then, using the other Bell state, Bob can teleport the qubit back to Alice. This approach uses two Bell states (one for each teleportation) and four classical bits of communication (two for each teleportation).

   The following diagram shows a simpler way of doing this, where they only need one Bell state and two classical bits of communication (one in each direction).

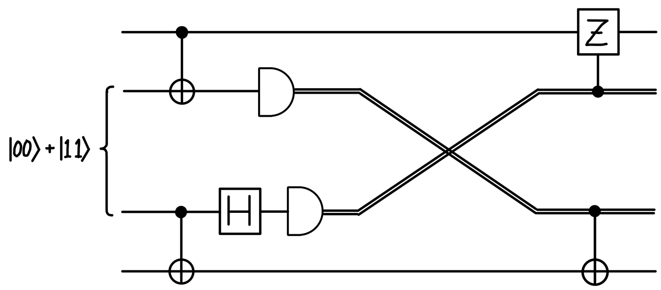
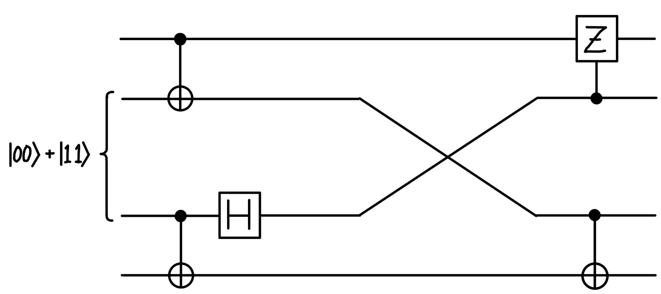


Figure 1: Simulation of a CNOT gate (between the first and last qubit)

   In the following, we analyze this circuit, in order to show that it works. We first consider this version of the circuit, where the measurement gates have been removed:



   (a) Show that the circuit without the measurements has the following property: For all computational basis states of the form $|a\rangle|b\rangle$ (where $a, b \in \{0,1\}$) the output of the circuit is $|a\rangle|b \oplus a\rangle$ and the final state of the two middle qubits is independent of a, b.

   (b) Explain why it follows from part (a) that the circuit without the measurements performs the CNOT gate for *all* 2-qubit input states. (For this to hold, it is important that, in part (a), the final state of the two middle qubits is independent of a, b.)

   (c) Explain why it follows from part (b) that the circuit in figure 1 performs the CNOT gate for all 2-qubit input states. (Hint: you may invoke the Deferred Measurement lemma, which is Lemma 8.1 in section 8.5 of the *Primer* lecture notes.)

4. **Determining multilinear functions [15 points].** For each $a_1, a_2, a_3 \in \{0,1\}$, define the function $f_{a_1,a_2,a_3} : \{0,1\}^3 \to \{0,1\}$ as

$$f_{a_1,a_2,a_3}(x_1, x_2, x_3) = a_1x_1 + a_2x_2 + a_3x_3 \bmod 2,$$

for all $x_1, x_2, x_3 \in \{0,1\}$.

(This is equivalent to defining $f_{a_1,a_2,a_3}(x_1, x_2, x_3) = (a_1 \wedge x_1) \oplus (a_2 \wedge x_2) \oplus (a_3 \wedge x_3)$.)

Let the goal be to determine a_1, a_2, a_3 (in other words, which of the eight possible functions f is).

(a) [6 points] What is the minimum number of classical queries required to solve this problem (with no error)? (Include a proof that it cannot be fewer.)

(b) [9 points] Show that one quantum query suffices to solve this problem (with no error). Include the algorithm and its analysis.