Assignment 1
Due date: 11:59pm, September 17, 2024

1. **Distinguishing between pairs of states [15 points; 5 for each part].** In each case, one of the two given states is prepared and sent to you (you are not told which one; each case arises with probability $\frac{1}{2}$). Your goal is to guess which of the two states it is. You are allowed to perform any measurement operation on the state to help you with this goal.

   Describe your distinguishing procedure as a unitary operation followed by a measurement (in the computational basis) and give its average-case success probability. (Your assigned grade will depend on how close your average-case success probability is to optimal.)

   (a) $|1\rangle$ vs. $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

   (b) $\frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$ vs. $\frac{1}{\sqrt{2}}|0\rangle + \frac{1-i}{2}|1\rangle$ (where $i = \sqrt{-1} = e^{i\pi/2}$)

   (c) $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ vs. $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$

2. **Applying 1-qubit gates to 2-qubit states [15 points; 5 each].** Let $\theta \in [0, 2\pi]$ and R_θ be the 2×2 rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{1}$$

   In each case, describe the resulting state after the operation is performed:

   (a) Apply R_θ to the *first* qubit of state $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$.

   (b) Apply R_θ to *both* qubits of state $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$.

   (c) Apply $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ to *both* qubits of state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ (where $i = \sqrt{-1}$).

3. **Entangled states and product states [15 points; 5 each].** For each two-qubit state below, either express it as a product of two one-qubit states or show that such a factorization is impossible (in the latter case, the qubits are *entangled*).

   (a) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

   (b) $\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

   (c) $\frac{1}{2}|00\rangle + \frac{1}{2}i|01\rangle - \frac{1}{2}i|10\rangle + \frac{1}{2}|11\rangle$

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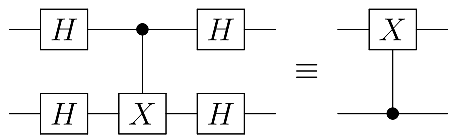
4. **Control-target inversion [15 points; 5 each].** Recall the three Pauli matrices

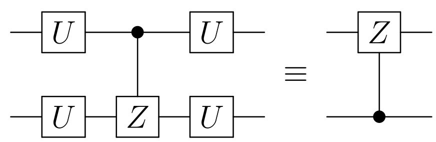
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(a) Prove that



where H is the Hadamard gate and the controlled-X gate is a CNOT gate.

(b) Give a 2×2 unitary operation U such that



(c) Give a 2×2 unitary operation V such that