Assignment 4
Due date: 11:59pm, November 14, 2023

1. **Probabilistic mixtures of Bell states [12 points; 4 each].**
Consider this probabilistic mixture of Bell states

$$\begin{cases} \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle & \text{with probability } \frac{1}{3}+\delta \\ \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle & \text{with probability } \frac{1}{3}-\delta \\ \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle & \text{with probability } \frac{1}{6} \\ \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle & \text{with probability } \frac{1}{6}, \end{cases} \tag{1}$$

where δ is a parameter such that $0 \leq \delta \leq \frac{1}{3}$.

(a) Give the 4×4 density matrix for this state. Write out the entries of this matrix.

(b) Give the density matrix for the special case where $\delta = 0$ (write out its entries). Is that state separable or entangled?
Give the density matrix for the special case where $\delta = \frac{1}{3}$ (write out its entries). Is that state separable or entangled?

(c) For what values of $\delta \in [0, \frac{1}{3}]$ is the state separable? Justify your answer.

Hint: You may use the positive partial transpose test to determine whether or not the state is separable.

2. **Action of unitary operations on the Bloch sphere [12 points; 4 each].**
For every 2×2 unitary matrix U, the effect of applying U on a qubit can be viewed as a rotation of the states in the Bloch sphere. For example, it can be shown that

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \tag{2}$$

corresponds to rotation by angle 2θ along the axis specified by state $|0\rangle$. In each case below, give the angle of rotation and also specify the axis of rotation:

(a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$.

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3. **Detail about measurements associated with Kraus operators [8 points].**
Prove that if matrices $A_1, A_2, \ldots, A_m$ are Kraus operators (i.e., they satisfy the equation $\sum_{k=1}^{m} A_k^* A_k = I$) and ρ is any valid density matrix then, for all $k \in \{1, 2, \ldots, m\}$ for which $\mathrm{Tr}(A_k \rho A_k^*) > 0$, it holds that

$$\frac{A_k \rho A_k^*}{\mathrm{Tr}(A_k \rho A_k^*)} \tag{3}$$

is also a valid density matrix. (A square matrix M is a valid density matrix if, for all states $|\psi\rangle$, it holds that $\langle\psi|M|\psi\rangle \geq 0$ and $\mathrm{Tr}(M) = 1$.)

Note that, in the context of a measurement, the instances of k for which $\mathrm{Tr}(A_k \rho A_k^*) = 0$ can be ignored because they never occur (i.e., they occur with probability 0).

4. **A particular class of quantum channels [12 points; 4 each].**
Define the qubit-to-qubit channels χ_A and χ_B as

$$\chi_A(\rho) = A_0 \rho A_0^* + A_1 \rho A_1^* \tag{4}$$

$$\chi_B(\rho) = B_0 \rho B_0^* + B_1 \rho B_1^*, \tag{5}$$

where

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} 0 & \sqrt{1-p} \\ 0 & 0 \end{bmatrix} \tag{6}$$

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{q} \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} 0 & \sqrt{1-q} \\ 0 & 0 \end{bmatrix}, \tag{7}$$

for parameters $p, q \in [0, 1]$.

(a) Confirm that A_0 and A_1 are valid Kraus operators by calculating

$$A_0^* A_0 + A_1^* A_1. \tag{8}$$

(b) Describe in words what the channel χ_A does in the case where $p = 0$. Describe in words what the channel χ_A does in the case where $p = 1$.

(c) Consider the channel obtained by first applying χ_A and then applying χ_B. Denote this composed channel as $\chi_B \circ \chi_A$, where $(\chi_B \circ \chi_A)(\rho) = \chi_B(\chi_A(\rho))$. From the definition, the composed channel $\chi_B \circ \chi_A$ has Kraus operators $B_0 A_0$, $B_0 A_1$, $B_1 A_0$, and $B_1 A_1$. Write out the entries of these four matrices. Is $\chi_B \circ \chi_A$ the same channel as χ_C, with Kraus operators

$$C_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{pq} \end{bmatrix} \quad \text{and} \quad C_1 = \begin{bmatrix} 0 & \sqrt{1-pq} \\ 0 & 0 \end{bmatrix}? \tag{9}$$

State your answer and justify it.

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5. **Some mixed-state state distinguishing problems [8 points, 4 each].**
Consider the following state distinguishing problems, where you are given the state on the left or the right (each, with probability $\frac{1}{2}$) and your goal is to correctly guess which of the two states it is. In each case, give the maximum possible success probability possible and describe the measurement procedure that attains it:

(a) state $|0\rangle$ vs. state $\begin{cases} |0\rangle & \text{with prob. } \frac{1}{2} \\ |1\rangle & \text{with prob. } \frac{1}{2} \end{cases}$

(b) state $|0\rangle$ vs. state $\begin{cases} |0\rangle & \text{with prob. } \frac{1}{2} \\ |+\rangle & \text{with prob. } \frac{1}{2} \end{cases}$

6. **Mixed unitary channels [8 points].**
A channel is *mixed-unitary* if it can be expressed in terms of Kraus operators $A_0, \cdots, A_{m-1}$ that are of the form $A_k = \sqrt{p_k}\, U_k$, where $U_0, \ldots, U_{m-1}$ are *unitary* and $(p_0, \ldots, p_{m-1})$ is a probability vector.

Consider the one-qubit channel that maps every 2×2 density matrix to the density matrix $\left[\begin{smallmatrix} 1/2 & 0 \\ 0 & 1/2 \end{smallmatrix}\right]$. Show that this is a mixed-unitary channel by exhibiting *four* Kraus operators, A_0, A_1, A_2, A_3, for it that are in the form of a mixed-unitary channel.

7. **(This is an optional question for bonus credit)**
Mixed unitary channels [6 points].
Prove that the channel in question 6 cannot be expressed as a mixed-unitary channel with only *three* Kraus operators.

Note: If you submit a solution to this question then there is a size-limit of one page.