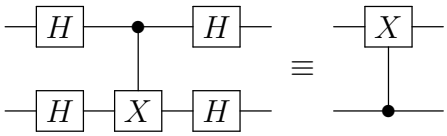
Assignment 2
Due date: 11:59pm, September 28, 2021

1. **Control-target inversion.** Recall the three Pauli matrices

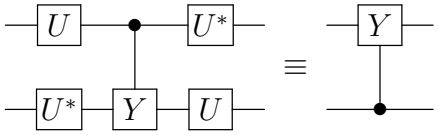
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   (a) **[4 points]** Prove that

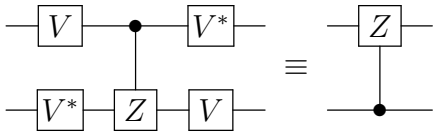


   where H is the Hadamard gate and the controlled-X gate is a CNOT gate.

   (b) **Optional bonus question [5 extra points]** Give a 2×2 unitary U such that



   (c) **[4 points]** Give a 2×2 unitary operation V such that



2. **Rotation around an axis in three dimensions [12 points; 4 each].** Consider a 1-qutrit space, with computational basis states $|0\rangle$, $|1\rangle$, $|2\rangle$.

   (a) Write down a 3×3 matrix corresponding to rotating by angle $\pm\theta$ around the axis $|2\rangle$. ($\pm\theta$ because of the ambiguity about rotation direction; either direction is fine.)

   (b) Write down a 3×3 matrix corresponding to rotating by angle $\pm\theta$ around the axis $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.

   (c) Write down a 3×3 matrix corresponding to rotating by angle $\pm\theta$ around the axis $\frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle$.

3. **Distinguishing between a set of "tetrahedral" states [10 points].** Consider the following four 2-qubit states (note that they are confined to a 3-dimensional space):

$$|\psi_0\rangle = \tfrac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle) \qquad |\psi_2\rangle = \tfrac{1}{\sqrt{3}}(-|00\rangle + |01\rangle - |10\rangle)$$
$$|\psi_1\rangle = \tfrac{1}{\sqrt{3}}(|00\rangle - |01\rangle - |10\rangle) \qquad |\psi_3\rangle = \tfrac{1}{\sqrt{3}}(-|00\rangle - |01\rangle + |10\rangle)$$

   It's easy to check that the inner product between each pair is $-1/3$, and they can be viewed geometrically as corresponding to the vertices of a regular tetrahedron in three dimensions. Suppose one of these states is selected and sent to you, and your goal is to guess which state it is. Describe a procedure based on unitary operations and measurements on the two-qubit system that predicts the state with as high a worst-case success probability as you can achieve. (Your grade will depend on how close your procedure is to optimal.)