Assignment 8

Due: 11:59pm, Tuesday, November 23, 2021 (note change)

1. **Unitary operations as rotations on the Bloch sphere [12 points, 6 each].** It is known that every one-qubit unitary operation acts on the Bloch sphere as a rotation about some axis. For each of the following unitary operations, describe the axis of rotation (which can be an eigenvector of the unitary) and the angle of rotation on the Bloch sphere:

   (a) $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$

   (b) $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$

2. **Square roots of channels [18 points; 9 each].** For any quantum channel mapping qubits to qubits, $\mathcal{C} : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$, it makes sense to compose $\mathcal{C}$ with itself, which means apply the channel $\mathcal{C}$ twice in succession. A *square root* of a channel $\mathcal{C}$ is a channel $\mathcal{B}$ such that $\mathcal{B}$ composed with itself is $\mathcal{C}$.

   (a) [9 points] Give a square root of the channel specified by these Kraus operators:

   $$A_0 = \begin{bmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{p} \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} 0 & \sqrt{1-p} \\ \sqrt{1-p} & 0 \end{bmatrix}, \tag{1}$$

   assuming that p is an arbitrary real-valued parameter such that $\frac{1}{2} \le p < 1$.
   (And [3 bonus points] if you also cover the case where $0 < p < \frac{1}{2}$. *Please only answer this bonus part if you're confident that you have a solution and a clear exposition of it.*)

   (b) [9 points] Give a square root of the channel specified by these Kraus operators:

   $$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}, \tag{2}$$

   assuming that p is an arbitrary real-valued parameter such that $0 < p < 1$.

   (c) [6 bonus points] Does every channel that maps qubits to qubits (i.e., maps a 1-qubit input to a 1-qubit output) have a square root? Justify your answer. *Please only answer this bonus part if you're confident that you have a solution and a clear exposition of it.*

3. **(This is an optional question for bonus credit) [8 points].**
   Consider the "maximally mixing channel" for qubits, which maps every qubit state ρ to the maximally mixed state $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$. Show that this channel cannot be implemented with three Kraus operators.