Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lectures 19-20 (2019)

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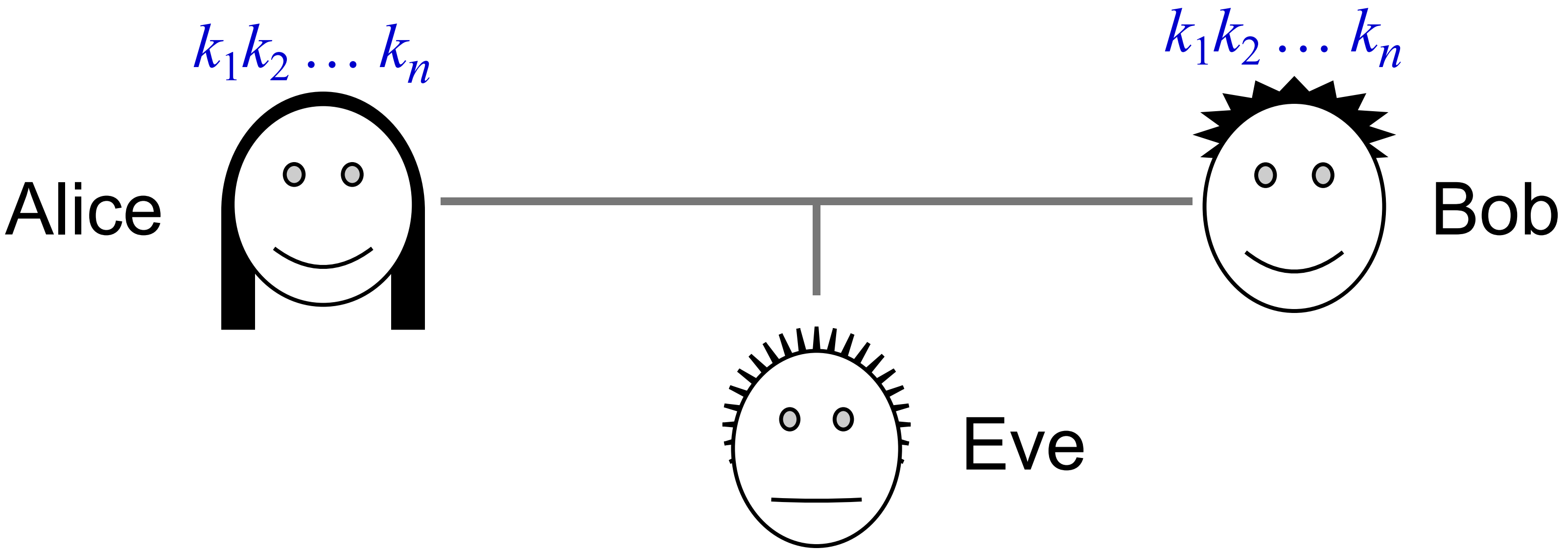
Quantum key distribution

Private communication



$k_1k_2 \ldots k_n$

$k_1k_2 \ldots k_n$

Alice

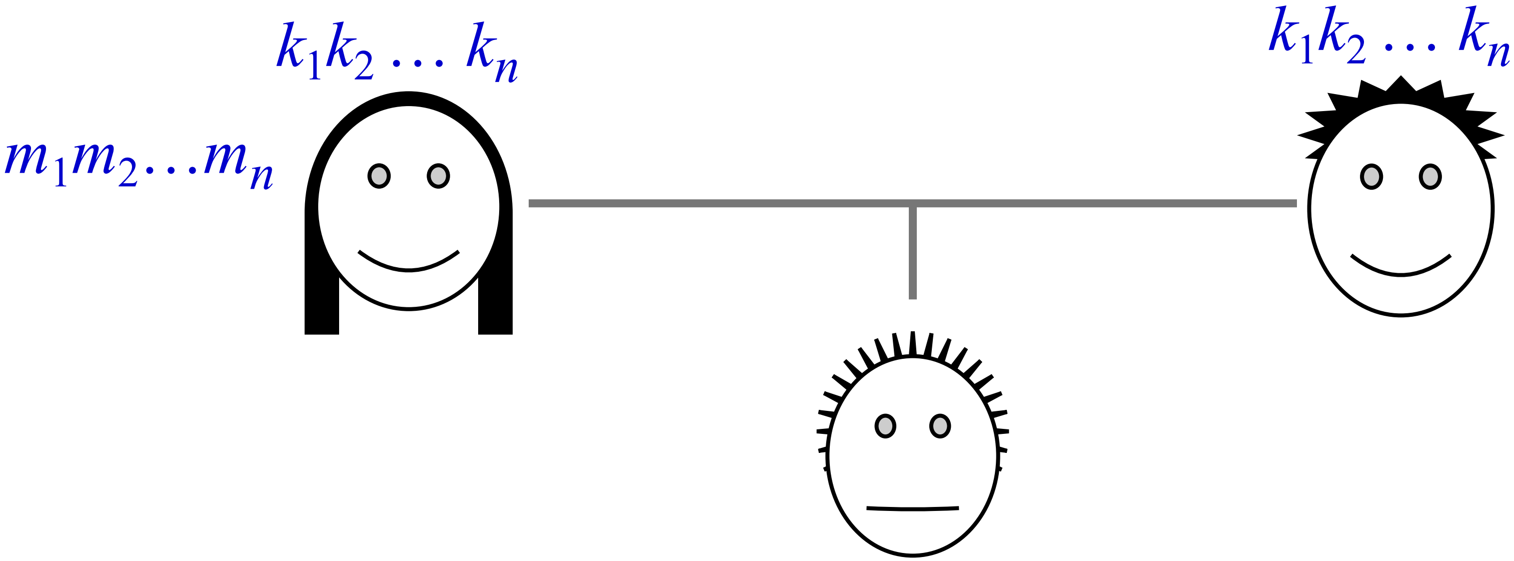
Bob

Eve

Scenario: Alice and Bob would like to communicate privately in the presence of an eavesdropper Eve

- A provably secure (classical) scheme exists for this, the ***one-time pad***
- The one-time pad requires Alice & Bob to share a ***secret key***: $k \in \{0,1\}^n$, uniformly distributed (secret from Eve)

Private communication



One-time pad protocol:

- Alice sends $c = m\oplus k$ (the bit-wise $\oplus$) to Bob
- Bob computes $c\oplus k$, which is $(m\oplus k)\oplus k = m$

This is secure because, what Eve sees is c, and c is uniformly distributed, regardless of what m is

But how do Alice and Bob set up the secret key to begin with?

Key distribution scenario

Key distribution problem: set up a large number of secret key bits

Note: for security, Alice and Bob must never reuse the key bits

E.g., if Alice encrypts both m and m' using the same key k then Eve can deduce $m \oplus m' = c \oplus c'$

Simple, but cumbersome approaches:

- Alice and Bob get together and flip coins
- Alice and Bob use a trusted third party

Key distribution based on computational hardness

Public key cryptosystems (e.g., RSA)

Bob generates two keys:

- a ***public key*** (for encryption)
- a ***secret (private) key*** (for decryption)

Since the decryption function is essentially the inverse of the encryption, in principle, it is possible to decrypt using only the public key; however, decryption using only the public key is (presumed) computationally hard (functions with this property are called ***trapdoor one-way functions***)

Using a public key cryptosystem, Alice can choose the key, encrypt it using Bob's public key and send it to Bob (who can then decrypt the message)

The security of many such schemes is based on the presumed computational difficulty of factoring integers—hence breakable with quantum computers!

Other schemes (e.g., elliptic curve cryptography schemes) are also breakable by quantum computers

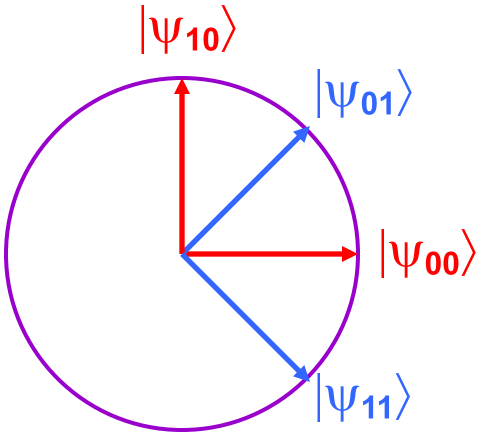
Quantum key distribution (QKD)

- A protocol that enables Alice and Bob to set up a secure* secret key, provided that they have:
  - A ***quantum channel***, where Eve can read and modify messages
  - An ***authenticated classical channel***, where Eve can read messages, but cannot tamper with them (the authenticated classical channel can be simulated by Alice and Bob having a ***very short*** classical secret key)

- There are several protocols for QKD, and the first one proposed is called "**BB84**" [Bennett & Brassard, 1984]:
  - BB84 is "easy to implement" physically, but "difficult" to prove secure
  - [Mayers, 1996]: first true security proof (quite complicated)
  - [Shor & Preskill, 2000]: "simple" proof of security

*** information-theoretic security**

BB84



- First, define: $|\psi_{00}\rangle = |0\rangle$

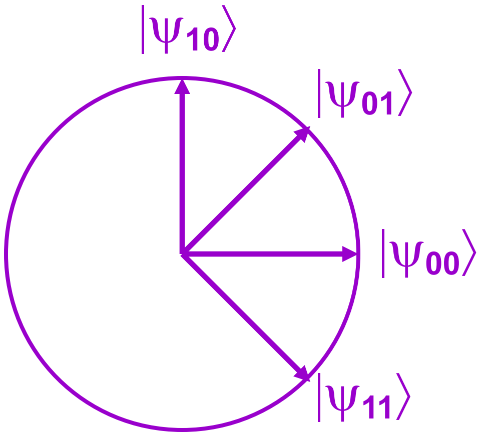
  $|\psi_{10}\rangle = |1\rangle$

  $|\psi_{11}\rangle = |-\rangle = |0\rangle - |1\rangle$

  $|\psi_{01}\rangle = |+\rangle = |0\rangle + |1\rangle$

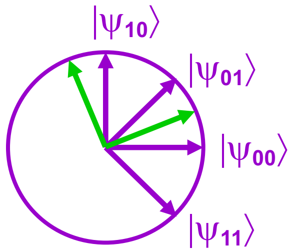
- Alice begins with two random n-bit strings $a, b \in \{0,1\}^n$
- Alice sends the state $|\psi\rangle = |\psi_{a_1b_1}\rangle|\psi_{a_2b_2}\rangle \dots |\psi_{a_nb_n}\rangle$ to Bob
- **Note:** Eve may see these qubits (and tamper with them)
- After receiving $|\psi\rangle$, Bob randomly chooses $b' \in \{0,1\}^n$ and measures each qubit as follows:
  - If $b'_i = 0$ then measure qubit in basis $\{|0\rangle, |1\rangle\}$, yielding outcome a'_i
  - If $b'_i = 1$ then measure qubit in basis $\{|+\rangle, |-\rangle\}$, yielding outcome a'_i

BB84



- **Note:**
  - If $b'_i = b_i$ then $a'_i = a_i$
  - If $b'_i \neq b_i$ then $\Pr[a'_i = a_i] = ½$
- Bob informs Alice when he has performed his measurements (using the public channel)
- Next, Alice reveals b and Bob reveals b' over the public channel
- They discard the cases where $b'_i \neq b_i$ and they will use the ***remaining bits*** of a and a' to produce the key
- **Note:**
  - If Eve did not disturb the qubits then the key can be just a $(= a')$
  - The ***interesting*** case is where Eve may tamper with $|\psi\rangle$ while it is sent from Alice to Bob

BB84



- **Intuition:**
  - Eve cannot acquire information about $|\psi\rangle$ without disturbing it, which will cause ***some*** of the bits of a and a' to disagree
  - It can be proven* that: the more information Eve acquires about a, the more bit positions of a and a' will be different
- From Alice and Bob's remaining bits, a and a' (where the positions where $b'_i \neq b_i$ have already been discarded):
  - They take a random subset and reveal them in order to estimate the fraction of bits where a and a' disagree
  - If this fraction is not too high then they proceed to distill a key from the bits of a and a' that are left over (around $n/4$ bits)

* To prove this rigorously is nontrivial

BB84

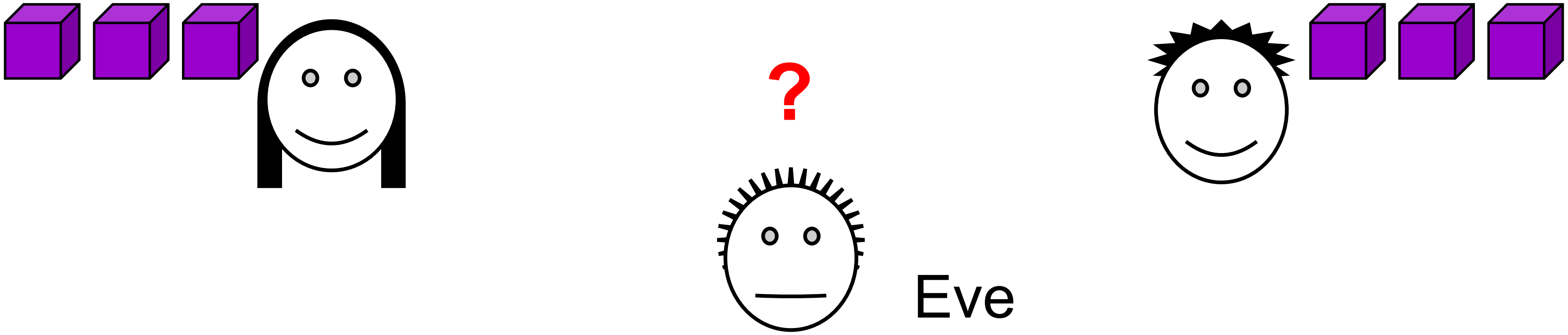
- If the error rate between a and a' is below some threshold (around 11%) then Alice and Bob can produce a good key using techniques from classical cryptography:
  - **Information reconciliation** ("distributed error correction") produces shorter a and a' such that**:**
    - (i) $a = a'$, and
    - (ii) Eve doesn't acquire much information about a and a' in the process
  - **Privacy amplification**
    produces shorter a and a' such that Eve's information about a and a' is small

- There are already commercially available implementations of BB84, though assessing their true security is a subtle matter
  (since their physical mechanisms are not ideal)

The Lo-Chau key exchange protocol: easier to analyze, though harder to implement

Sufficiency of Bell states

If Alice and Bob can somehow generate a series of Bell states between them, such as $|\phi^+\rangle|\phi^+\rangle...|\phi^+\rangle$, (where $|\phi^+\rangle = |00\rangle + |11\rangle$) then it suffices for them to measure these states to obtain a secret key



Intuitively, this is because there is nothing that Eve can "know" about $|\phi^+\rangle = |00\rangle + |11\rangle$ that will permit her to predict a future measurement that she has no access to

Key distribution protocol based on $|\phi^+\rangle$

Preliminary idea: Alice creates several $|\phi^+\rangle$ states and sends the second qubit of each one to Bob

If they knew that that they possessed state $|\phi^+\rangle|\phi^+\rangle \dots |\phi^+\rangle$ then they could simply measure each qubit pair (say, in the computational basis) to obtain a shared private key

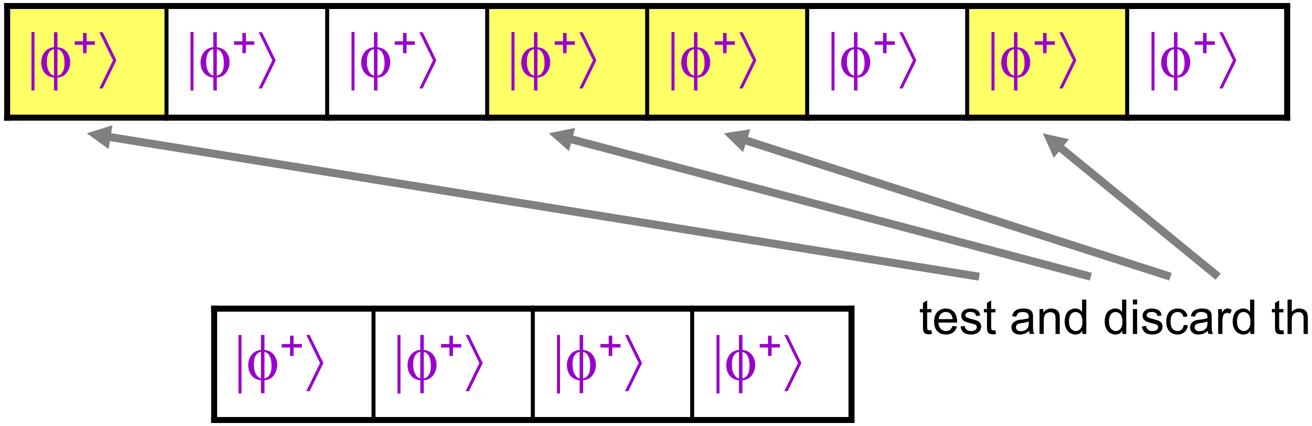
Since Eve can access the qubit channel, she can measure, or otherwise disturb the state in transit (e.g., collapse to $|00\rangle$ or $|11\rangle$, known to her)

We might as well assume that Eve is supplying the qubits to Alice and Bob, who somehow test whether they're $|\phi^+\rangle$

Question: how can Alice and Bob test the validity of their states?

Testing $|00\rangle + |11\rangle$ states (1)

Alice and Bob can pick a ***random subset*** of their $|\phi^+\rangle$ states (say, half of them) to test, and then forfeit those



test and discard these pairs

Question: How do Alice and Bob "test" the pairs in this subset?

Due to Eve, they cannot use the quantum channel to actually measure them in the Bell basis ... but they can do individual measurements and compare results via the classical channel

Testing $|00\rangle + |11\rangle$ states (2)

The Bell state $|\phi^+\rangle = |00\rangle + |11\rangle$ has the following properties:

(a) if both qubits are measured in the ***computational basis*** the resulting bits will be the same (i.e., 00 or 11)

(b) if both qubits are measured in the ***Hadamard basis*** the resulting bits will still be the same (since if $H \otimes H\,|\phi^+\rangle = |\phi^+\rangle$)

Moreover, $|\phi^+\rangle$ is the ***only*** two-qubit state that satisfies both properties **(a)** ***and*** **(b)**

Question: Why?

Testing $|00\rangle + |11\rangle$ states (3)

Problem: they can only measure in ***one*** of these two bases

Solution: they pick the basis randomly among the two types (Alice decides by flipping a coin and announcing the result to Bob on the read-only classical channel)

Example: if Eve slips in a state $|00\rangle$ and if Alice & Bob measure this pair in the Hadamard basis, they get the ***same*** bit with probability only ½ (so this cheating is detected with probability ¼)

Basis:	**computational**	**Hadamard**
	$a \oplus b$	$a \oplus b$
$\|\phi^+\rangle$	0	0
$\|\phi^-\rangle$	0	1
$\|\psi^+\rangle$	1	0
$\|\psi^-\rangle$	1	1

If Eve slips in $|\mu\rangle$ in place of $|\phi^+\rangle$ then the probability of ***failing*** the test is

$$\geq \frac{1 - \langle \mu | \phi^+ \rangle^2}{2}$$

For $|00\rangle = \frac{1}{\sqrt{2}}|\phi^+\rangle + \frac{1}{\sqrt{2}}|\phi^-\rangle$ this is ¼

Testing $|00\rangle + |11\rangle$ states (4)

Suppose there are n purported $|\phi^{+}\rangle$ states and Alice and Bob test m of them (and are left with $n-m$ key bits)

Suppose Eve slips in just one $|00\rangle$ state

Then the probability of this causing the test to fail (thereby ***detecting*** Eve) is only $m/4n$

Consider the extreme case, where Alice and Bob set $m = n-1$ (i.e., they test all but one), so the detection probability is $(n-1)/4n = 1/4(1-1/n) \leq 1/4$

Even in this extreme case, Eve can control the value of one key bit (without her being detected) with probability at least $3/4$

There is a much better approach …

Better testing (1)

Think of a related (simpler) classical problem: detect if a binary array contains at least one 1

0	0	0	0	1	0	0	0

If one is confined to examining ***individual bits***, this is difficult to do with very high probability making few tests

Suppose we have a primitive operation that tests the ***parity of any subset of bits***

Then the following procedure exposes a 1 with probability ½:

pick a random $r \in \{0,1\}^n$ and test if $r \cdot x = 0$

If $x \neq 00...0$ then this test detects this with probability ½

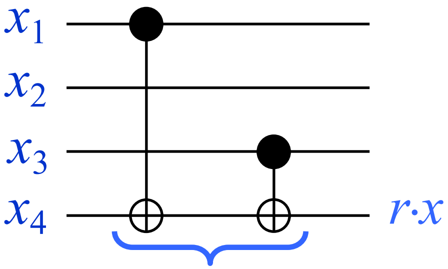
Testing k such parities detects with probability $1 - (½)^k$

Better testing (2)

Another way of interpreting this idea is to allow CNOT gates to be applied before a bit position is checked/discarded

Construct a circuit of CNOT gates in the following way:

choose a random $r \in \{0,1\}^n$ and compute $r{\cdot}x$ in some bit position using CNOT gates



example of circuit for $r = 1011$

Detects $x \neq 0000$ with probability ½ by only discarding one bit

In general, repeating this k times, detects with probability $1-(½)^k$ while only discarding k bits

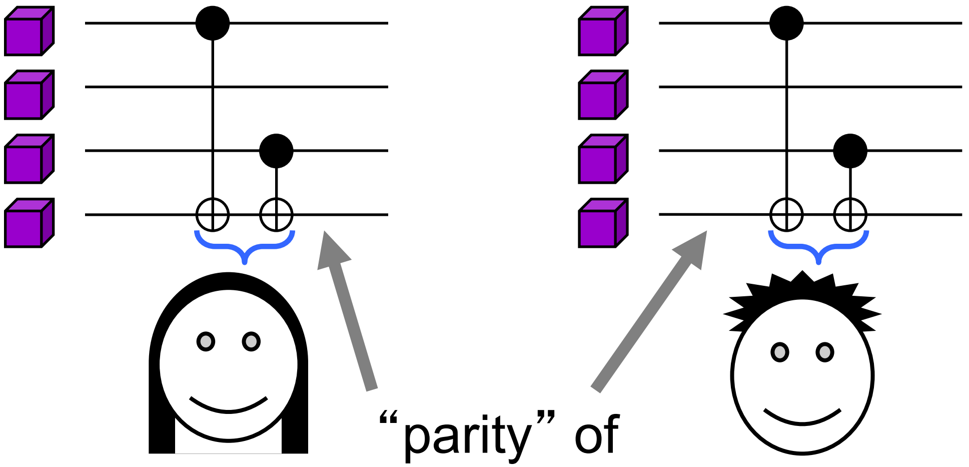
Methodology of bilateral CNOTS (1)

The previous idea can be translated into the context of testing whether pairs Bell states are all $|\phi^+\rangle$ or not

$\lvert\phi^+\rangle$	$\lvert\phi^+\rangle$	$\lvert\phi^+\rangle$	$\lvert\phi^+\rangle$	$\lvert\phi^+\rangle$	$\lvert\phi^+\rangle$	$\lvert\phi^+\rangle$	$\lvert\phi^+\rangle$

1. Alice picks a random $r \in \{0,1\}^n$ and sends it to Bob
2. Alice and Bob perform various bilateral CNOT operations on their qubits

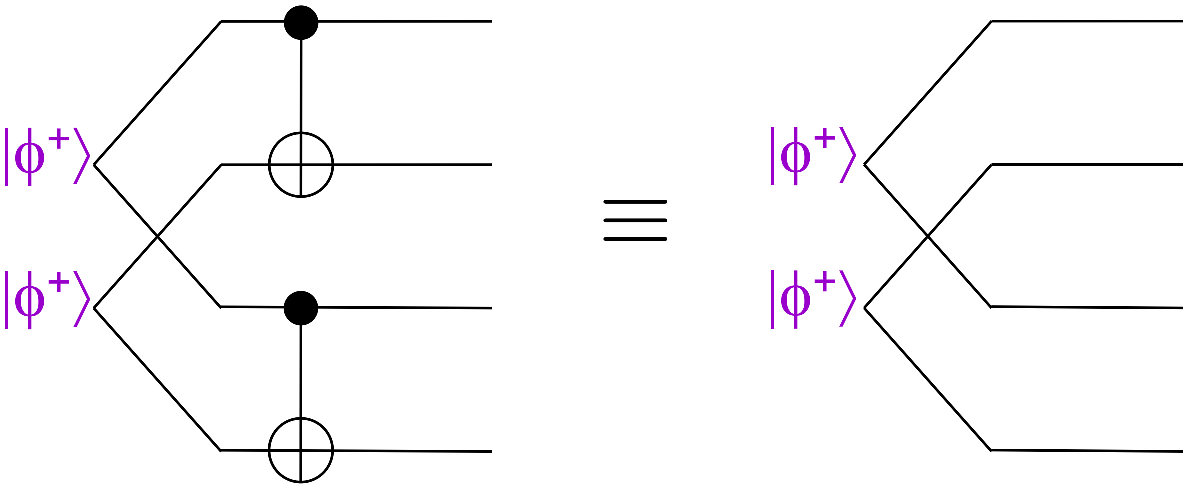
For $r = 1011$



“parity” of positions 1, 3, 4

Methodology of bilateral CNOTS (2)

Note that two $|\phi^+\rangle$ states remain unchanged when two CNOT gates are applied bilaterally across them as follows:

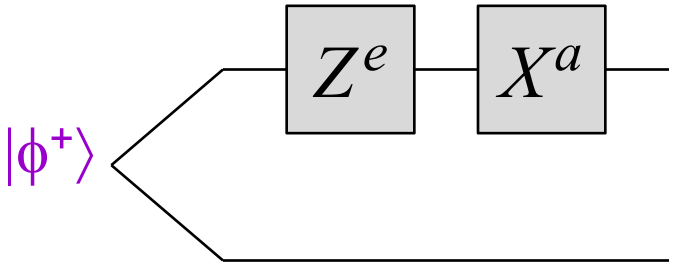


(This is a straightforward exercise to check)

Methodology of bilateral CNOTS (3)

Since $$X^a Z^e \otimes I\,|\phi^+\rangle = \begin{cases} |\phi^+\rangle = |00\rangle + |11\rangle & \text{if ae} = 00 \\ |\phi^-\rangle = |00\rangle - |11\rangle & \text{if ae} = 01 \\ |\psi^+\rangle = |10\rangle + |01\rangle & \text{if ae} = 10 \\ |\psi^-\rangle = |10\rangle - |01\rangle & \text{if ae} = 11 \end{cases}$$

we can think of each purported $|\phi^+\rangle$ state as: (where $a,\, e \in \{0,1\}^n$)

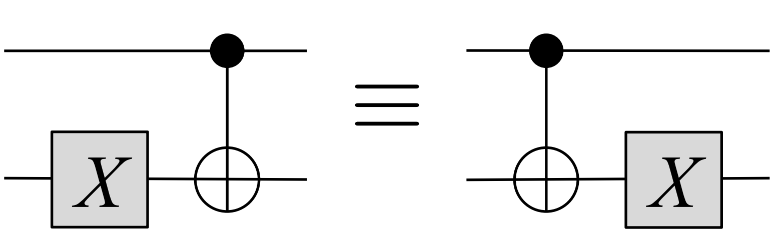


(We will consider ***general*** states—that are superpositions of states of the above form—later on)

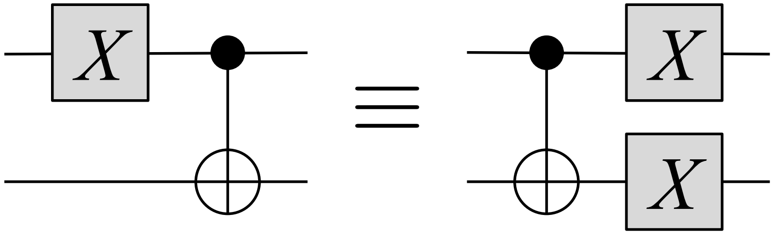
The goal is to determine detect if $a \neq 00...0$ or $e \neq 00...0$

Methodology of bilateral CNOTS (4)

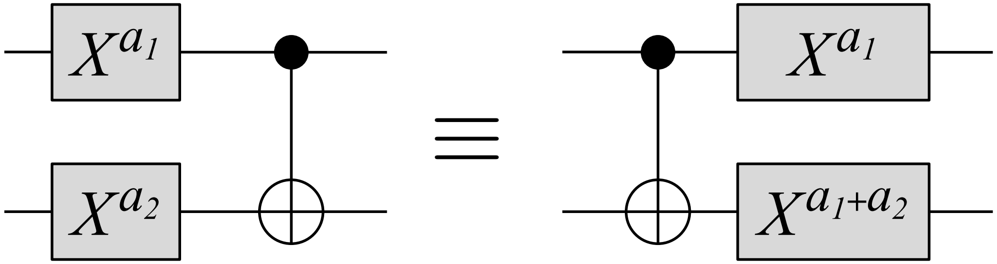
Note that:



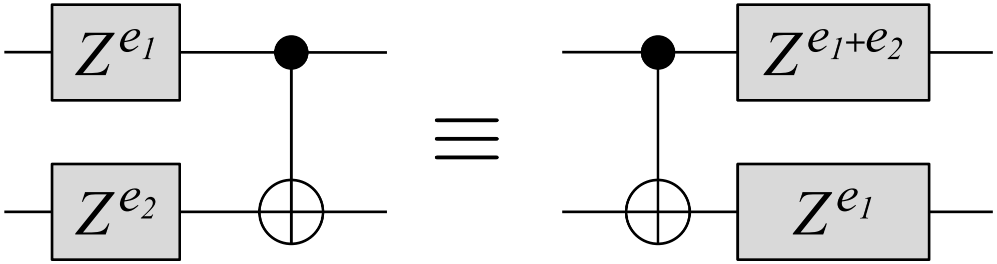
and



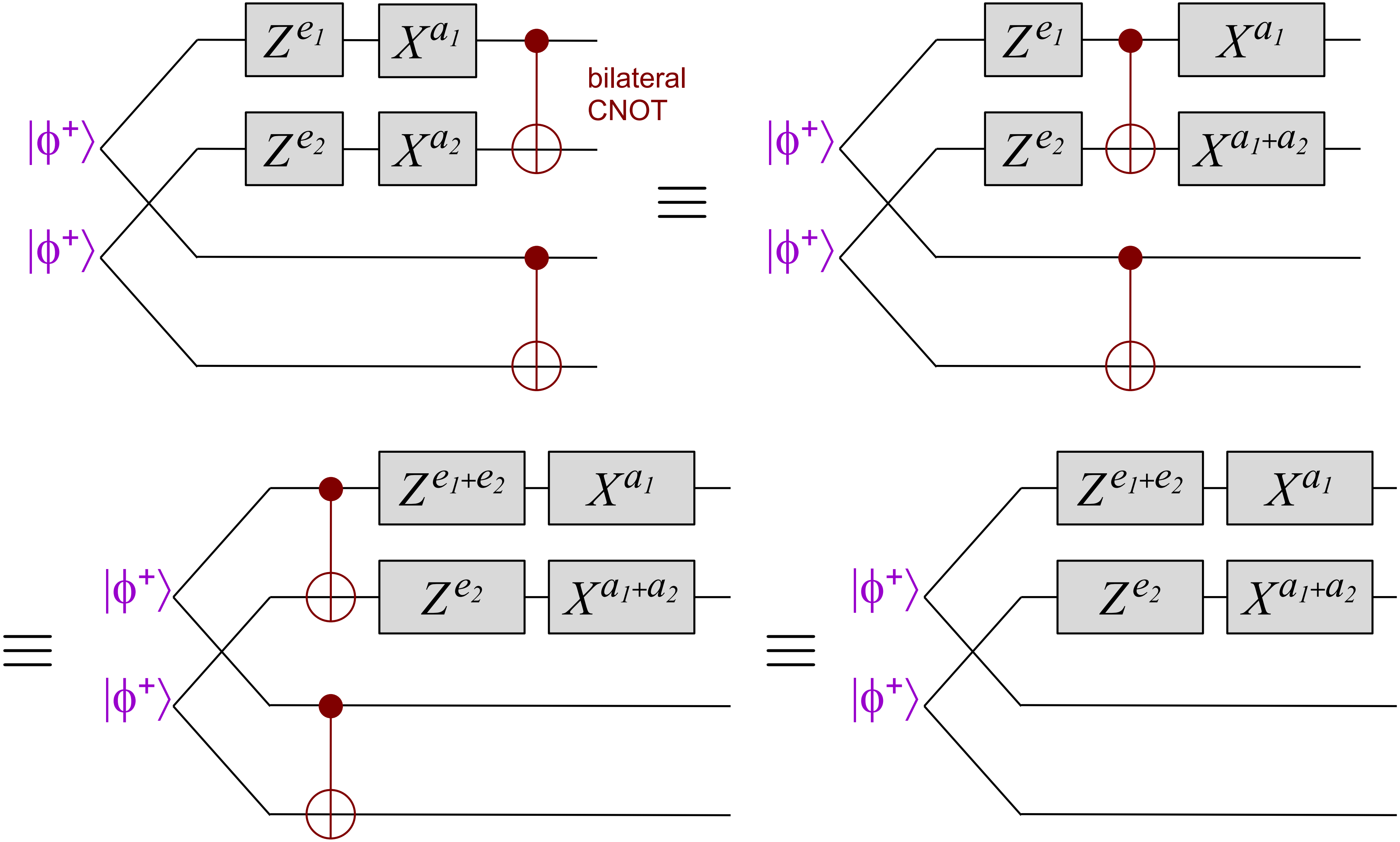
More generally:



Similarly:

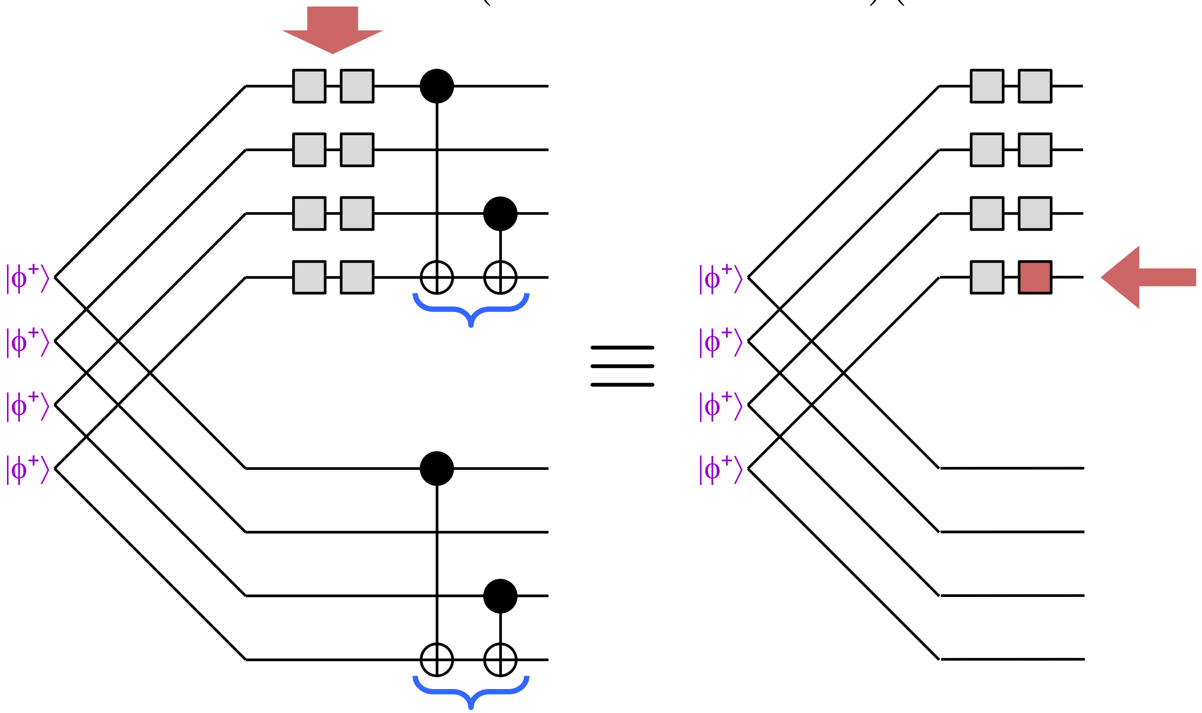


Methodology of bilateral CNOTS (5)



Methodology of bilateral CNOTS (6)

$$X^a Z^e = (X^{a_1} \otimes \cdots \otimes X^{a_n})(Z^{e_1} \otimes \cdots \otimes Z^{e_n})$$



$$X^{r \cdot a} = X^{r_1 a_1 + \cdots + r_n a_n}$$

This detects $a \neq 00...0$ with probability ½

This test in Hadamard basis detects $e \neq 00...0$ with probability ½

By randomly selecting which one of these two tests to perform, can detect **(**$a \neq 00...0$ ***or*** $e \neq 00...0$**)** with probability ¼

Methodology of bilateral CNOTS (7)

What happens if the process is repeated m times?

At iteration k:

Let the start state (for that iteration) be $X^aZ^e \otimes I\, |\phi^+\rangle^{\otimes n-(k-1)}$ for $a, e \in \{0,1\}^{n-(k-1)}$

The outcome (for that iteration) is either an end state or a decision to abort

Case 1: $a = e = 0\ldots0$ ("good" state)

Then Alice and Bob proceed, with end state is $|\phi^+\rangle^{\otimes n-k}$

Case 2: $a \neq 0\ldots0$ or $e \neq 0\ldots0$ ("bad" state)

Then and Alice and Bob abort the protocol with probability ¼

If A and B do not abort*: end state is $X^aZ^e \otimes I\, |\phi^+\rangle^{\otimes n-k}$ for $a, e \in \{0,1\}^{n-k}$

An ***attack*** is an initial state of the form $X^aZ^e \otimes I\, |\phi^+\rangle^{\otimes n}$ for $a, e \in \{0,1\}^n$

An attack ***succeeds*** if Alice and Bob do not abort and the final end state is $X^aZ^e \otimes I\, |\phi^+\rangle^{\otimes n-m}$ for $a, e \in \{0,1\}^{n-m}$ where $a \neq 0\ldots0$ or $e \neq 0\ldots0$

Claim: The probability that an attack succeeds after m rounds is $(¾)^m$

* Note: in case 2, a bad start state may become a good end state

Conclusion of Lo-Chau scheme

What if Eve provides a states that is not of the form

$X^a Z^e|\Phi^+\rangle = (X^{a_1} \otimes \cdots \otimes X^{a_n})(Z^{e_1} \otimes \cdots \otimes Z^{e_n})|\phi^+\rangle \otimes \cdots \otimes |\phi^+\rangle$?

Very rough idea: at every round the start state is a superposition of the form

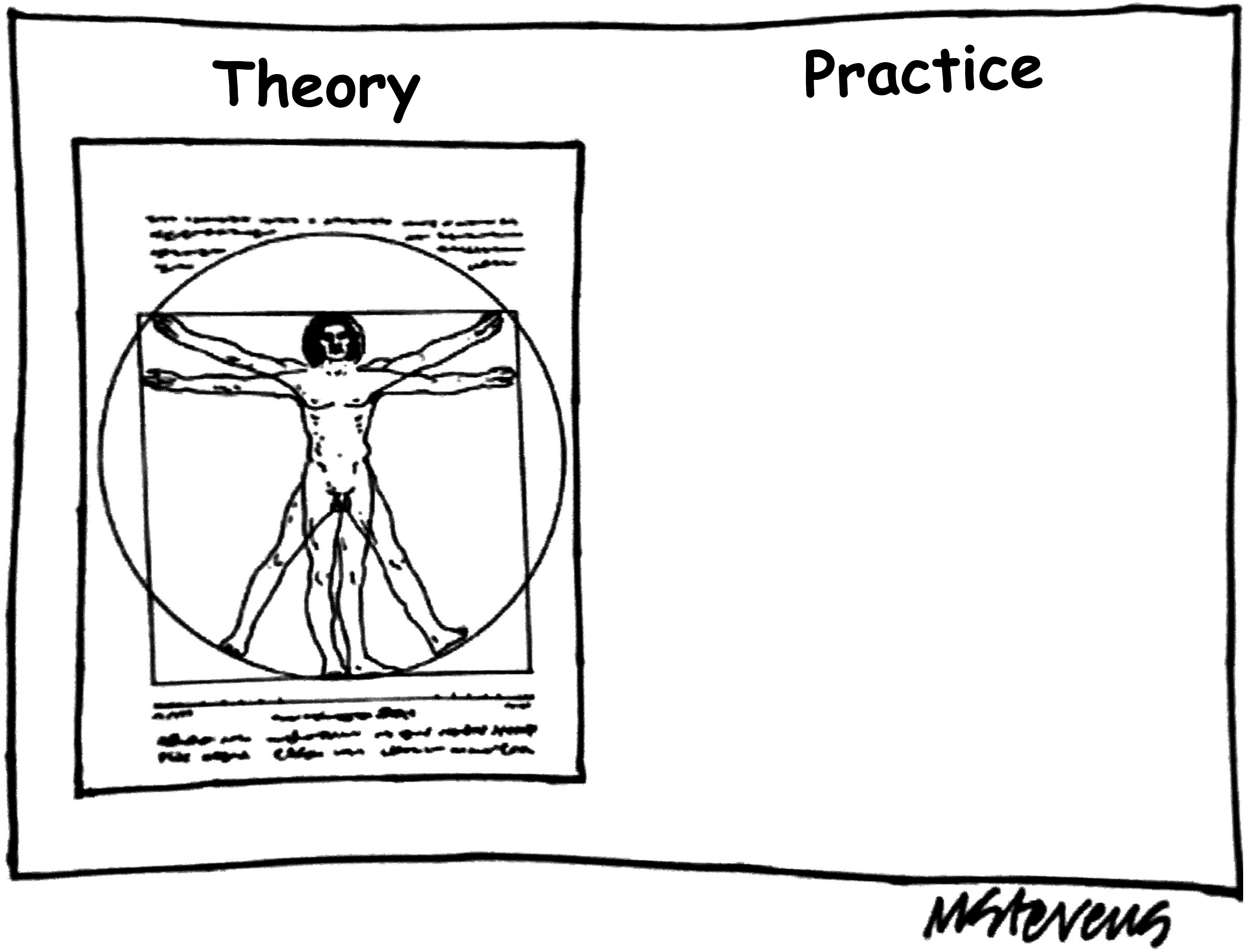
$|\mu\rangle = \sum_{a,e} \alpha_{a,e} X^a Z^e \otimes I|\Phi^+\rangle$ and the procedure aborts with prob. $\frac{1-\langle\mu|\Phi^+\rangle^2}{2}$

If $\langle\mu|\Phi^+\rangle^2$ is not close to 1 then the procedure has a good chance of aborting
if $\langle\mu|\Phi^+\rangle^2$ is close to 1 then Alice and Bob can safely use it in place of $|\Phi^+\rangle$ to generate their secret key

Sacrificing (say) half the qubit pairs, Alice and Bob can establish a key that Eve has exponentially small information about

Unlike BB84, this protocol requires Alice and Bob to have quantum computers —to store and perform nontrivial operations on states of several qubits

Theory vs. Practice in cryptography



(with apologies to M. Stevens)