Introduction to Quantum Information Processing

QIC 710 / CS 768 / PH 767 / CO 681 / AM 871

Lectures 13–14 (2019)

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Preliminary remarks about quantum communication

Quantum information can apparently be used to substantially reduce ***computation*** costs for a number of interesting problems

How does quantum information affect the ***communication costs*** of information processing tasks?

We explore this issue ...

Entanglement and signaling

Recall that Entangled states, such as $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$,

can be used to perform some intriguing feats, such as ***teleportation*** and ***superdense coding***

—but they ***cannot*** be used to "signal instantaneously"

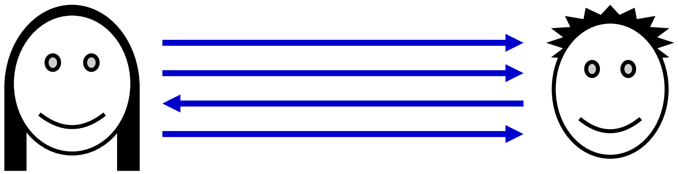
Any operation performed on one system has no affect on the state of the other system (its reduced density matrix)

Basic communication scenario

Goal: convey n bits from Alice to Bob

Basic communication scenario

Bit communication:

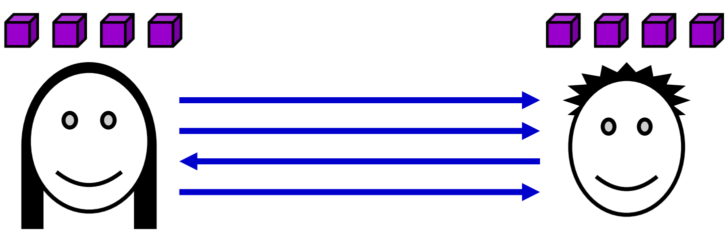


Cost: n

Qubit communication:

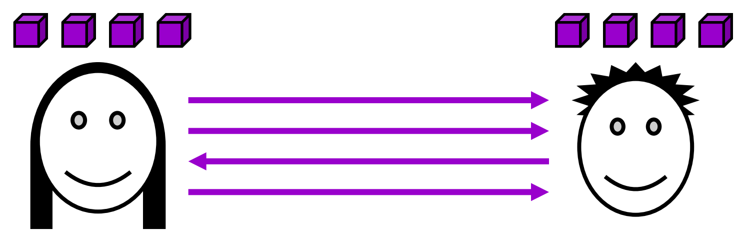
Cost: n [Holevo’s Theorem, 1973]

Bit communication & prior entanglement:



Cost: n (can be deduced)

Qubit communication & prior entanglement:



Cost: $n/2$ superdense coding

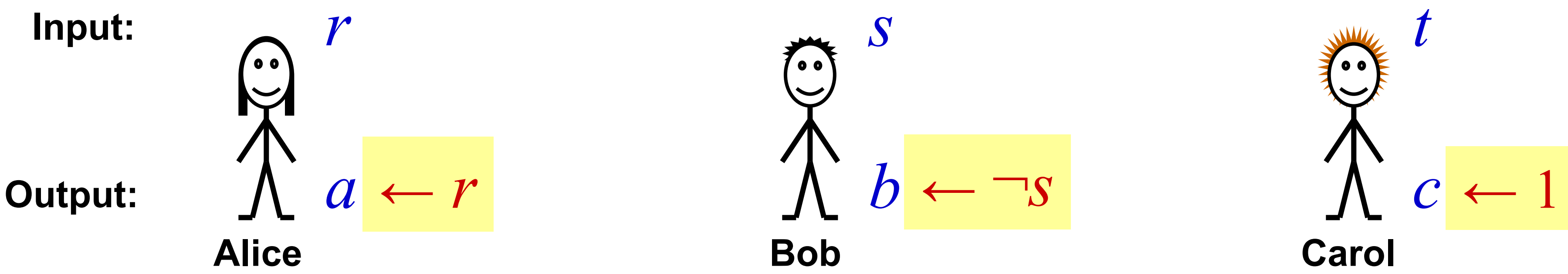
[Bennett & Wiesner, 1992]

The GHZ “paradox”

(Greenberger-Horne-Zeilinger)

GHZ scenario

[Greenberger, Horne, Zeilinger, 1980]



Rules of the game:

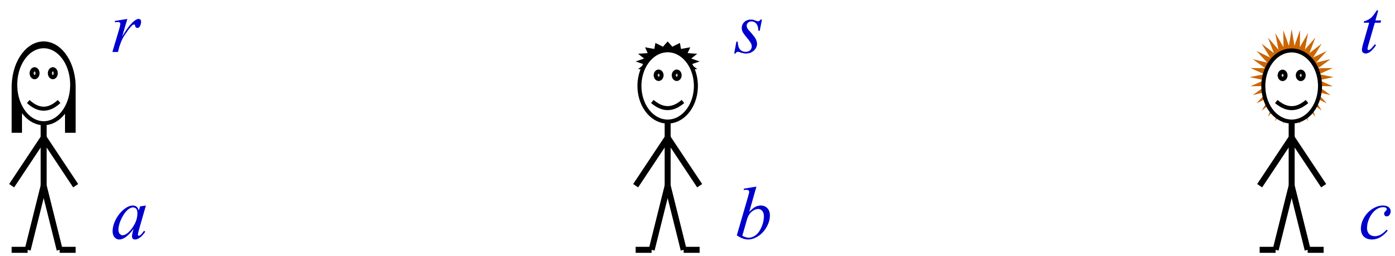
1. It is promised that $r \oplus s \oplus t = 0$
2. No communication after inputs received
3. They ***win*** if $a \oplus b \oplus c = r \vee s \vee t$

rst	$a \oplus b \oplus c$	abc
000	0	011
011	1	001
101	1	111
110	1	101

No perfect strategy for GHZ

Input:

Output:



rst	$a \oplus b \oplus c$
000	0
011	1
101	1
110	1

General deterministic strategy:

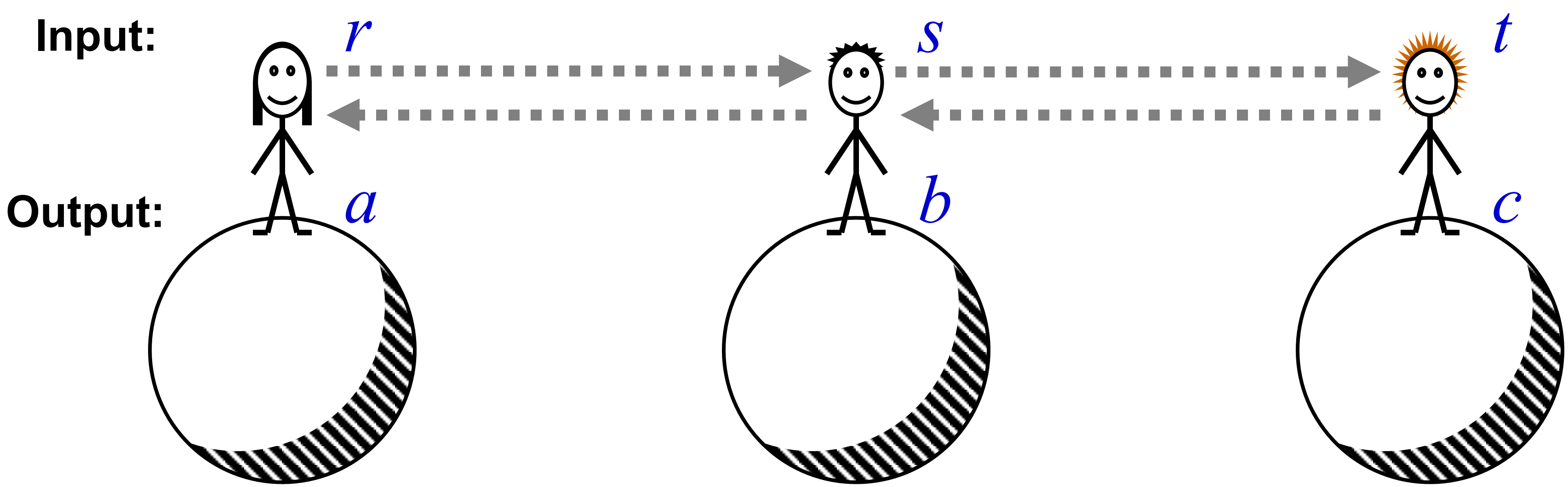
$a_0, a_1, b_0, b_1, c_0, c_1$

Winning conditions:

$$\begin{cases} a_0 \oplus b_0 \oplus c_0 = 0 \\ a_0 \oplus b_1 \oplus c_1 = 1 \\ a_1 \oplus b_0 \oplus c_1 = 1 \\ a_1 \oplus b_1 \oplus c_0 = 1 \end{cases}$$

Has no solution, thus no perfect strategy exists

GHZ: preventing communication



Input and output events can be ***space-like*** separated: so signals at the speed of light are not fast enough for cheating

What if Alice, Bob, and Carol ***still*** keep on winning?

To be continued …

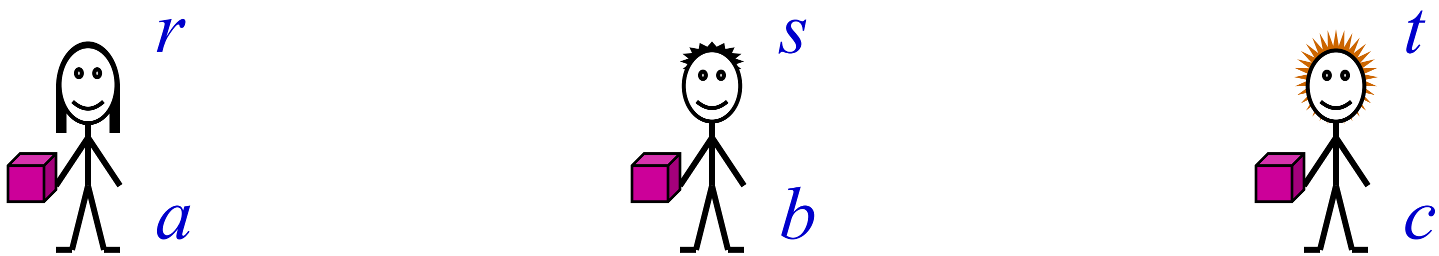
Continuation of:

The GHZ “paradox”

(Greenberger-Horne-Zeilinger)

"GHZ Paradox" explained

Prior entanglement: $|\psi\rangle = |000\rangle - |011\rangle - |101\rangle - |110\rangle$



<u>Alice's strategy:</u>

1. if $r = 1$ then apply H to qubit (else I)
2. measure qubit and set a to result

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

<u>Bob's & Carol's strategies:</u> similar

Case 1 ($rst = 000$)**:** state is measured directly …

Case 2 ($rst = 011$)**:** new state $|001\rangle + |010\rangle - |100\rangle + |111\rangle$

Cases 3 & 4 ($rst = 101$ & 110)**:** similar by symmetry

GHZ: conclusions

- For the GHZ game, any ***classical*** team succeeds with probability at most ¾
- Allowing the players to communicate would enable them to succeed with probability 1
- Entanglement cannot be used to communicate
- Nevertheless, allowing the players to have entanglement enables them to succeed with probability 1 (but not by using entanglement to communicate)
- Thus, entanglement is a useful resource for the task of ***winning the GHZ game***

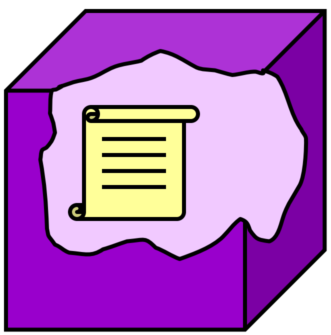
The Bell inequality and its violation
– Physicist's perspective

Bell’s Inequality and its violation

Part I: physicist’s view:

Can a quantum state have ***pre-determined*** outcomes for each possible measurement that can be applied to it?

qubit:



where the “manuscript” is something like this:

if $\{|0\rangle,|1\rangle\}$ measurement then output **0**

if $\{|+\rangle,|-\rangle\}$ measurement then output **1**

if **...** (etc)

called ***hidden variables***

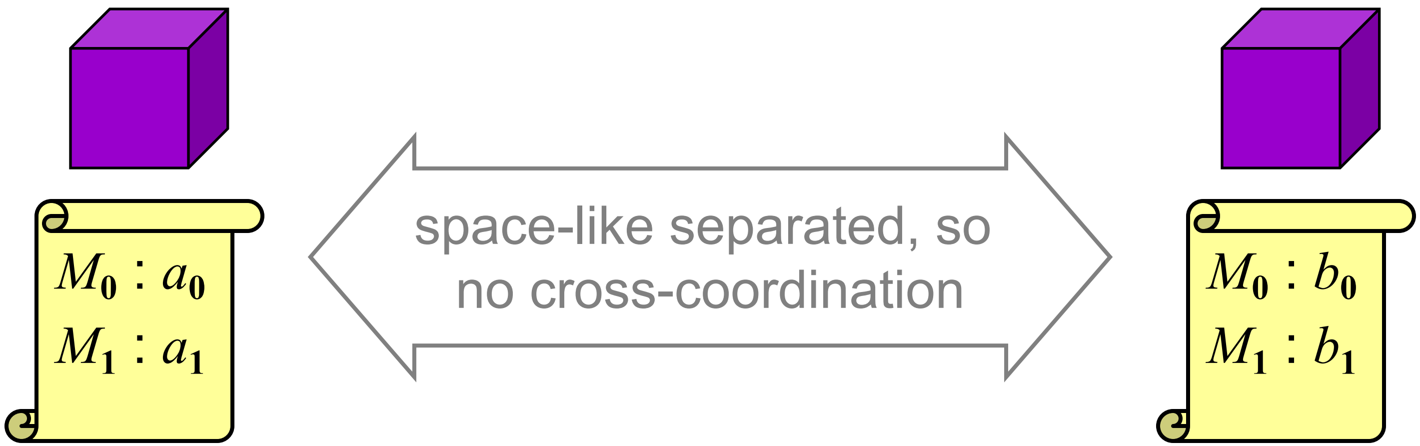
table could be implicitly given by some formula

[Bell, 1964]

[Clauser, Horne, Shimony, Holt, 1969]

Bell Inequality

Imagine a two-qubit system, where one of two measurements, called M_0 and M_1, will be applied to each qubit:



Define:

$A_0 = (-1)^{a_0}$
$A_1 = (-1)^{a_1}$
$B_0 = (-1)^{b_0}$
$B_1 = (-1)^{b_1}$

Claim: $A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$

Proof: $A_0 (B_0 + B_1) + A_1 (B_0 - B_1) \leq 2$

one is ± 2 and the other is 0

Bell Inequality

$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$ is called a ***Bell Inequality****

Question: could one, in principle, design an experiment to check if this Bell Inequality holds for a particular system?

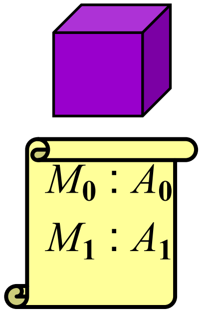
Answer 1: ***no, not directly***, because A_0, A_1, B_0, B_1 cannot all be measured (only ***one*** $A_s B_t$ term can be measured)

Answer 2: ***yes, indirectly***, by making many runs of this experiment: pick a random $st \in \{00, 01, 10, 11\}$ and then measure with M_s and M_t to get the value of $A_s B_t$

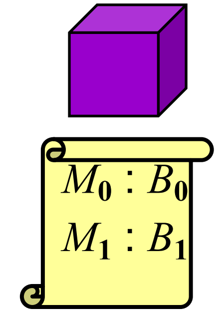
The ***average*** of $A_0 B_0,\ A_0 B_1,\ A_1 B_0,\ -A_1 B_1$ should be ≤ ½

* also called CHSH Inequality

Recap of Bell Inequality



Assume local hidden variables framework is correct



Consider the following experiment:

1. pick a random $st \in \{00, 01, 10, 11\}$ (uniform distribution)
2. perform M_s measurement on 1st qubit (outcome $A_s \in \{+1, -1\}$)
3. perform M_t measurement on 2nd qubit (outcome $B_t \in \{+1, -1\}$)
4. output the value of $(-1)^{s \cdot t} A_s B_t$

In any run of this experiment, the output is an element of $\{+1, -1\}$ (according to probabilities that depend on what A_0, A_1, B_0, B_1 are)

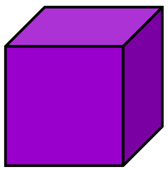
How large can the ***expected*** value of the outcome be?

$$\tfrac{1}{4}(A_0 B_0) + \tfrac{1}{4}(A_0 B_1) + \tfrac{1}{4}(A_1 B_0) + \tfrac{1}{4}(-A_1 B_1)$$

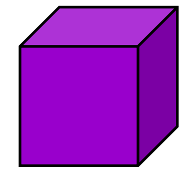
$$= \tfrac{1}{4}(A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1) \leq \tfrac{1}{4}\, 2 = \tfrac{1}{2}$$

Violating the Bell Inequality

Assume the quantum mechanical framework is correct



Two-qubit system in state
$|\phi\rangle = |00\rangle - |11\rangle$



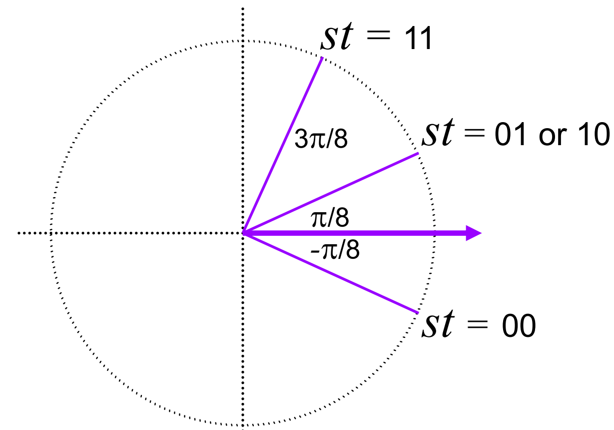
It can be shown that, applying rotations θ_A and θ_B ($R_{\theta_A} \otimes R_{\theta_B}$) yields:

$$\cos(\theta_A + \theta_B)\,\underbrace{(|00\rangle - |11\rangle)}_{AB\,=\,+1} + \sin(\theta_A + \theta_B)\,\underbrace{(|01\rangle + |10\rangle)}_{AB\,=\,-1}$$

Define

M_0: rotate by $-\pi/16$ then measure

M_1: rotate by $+3\pi/16$ then measure



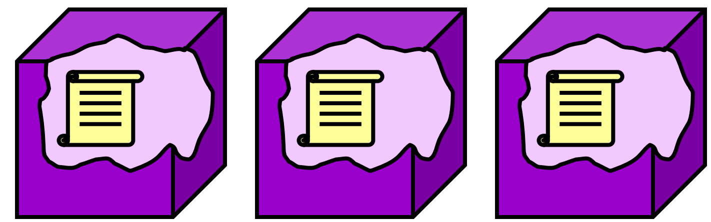
$\cos^2(\pi/8) = ½ + ¼\sqrt{2}$

Then $A_0 B_0$, $A_0 B_1$, $A_1 B_0$, $-A_1 B_1$ all have expected value $½\sqrt{2}$, which ***contradicts*** the upper bound of ½

Therefore, QM framework implies LHV framework is false

Bell Inequality violation: summary

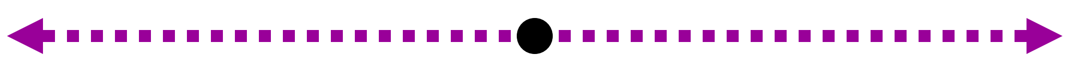
Assuming that quantum systems are governed by ***local hidden variables*** leads to the Bell inequality
$A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2$



But this is ***violated*** in the case of Bell states (by a factor of $\sqrt{2}$)

Therefore, no such hidden variables exist

This is, in principle, experimentally verifiable, and experiments along these lines have actually been conducted



The Bell inequality and its violation – Computer Scientist's perspective

Bell's Inequality and its violation

Part II: computer scientist's view:

input: s t

output: a b

Rules:
1. No communication after inputs received
2. They ***win*** if $a \oplus b = s \wedge t$

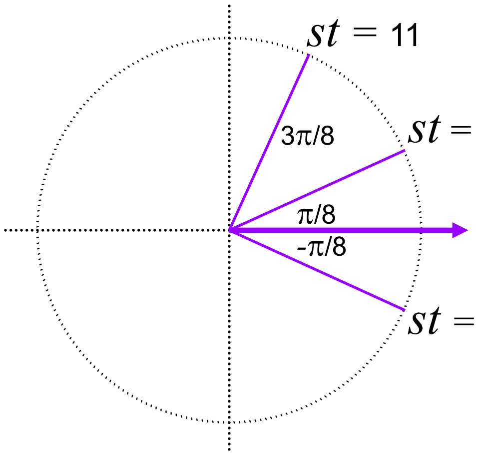
st	$a \oplus b$
00	0
01	0
10	0
11	1

With classical resources, $\Pr[a \oplus b = s \wedge t] \leq 0.75$

But, with prior entanglement state $|00\rangle - |11\rangle$,
$\Pr[a \oplus b = s \wedge t] = \cos^2(\pi/8) = ½ + ¼\sqrt{2} = 0.853\ldots$

The quantum strategy

- Alice and Bob start with entanglement
  $|\phi\rangle = |00\rangle - |11\rangle$
- **Alice:** if $s = 0$ then rotate by $\theta_A = -\pi/16$
  else rotate by $\theta_A = +3\pi/16$ and measure
- **Bob:** if $t = 0$ then rotate by $\theta_B = -\pi/16$
  else rotate by $\theta_B = +3\pi/16$ and measure

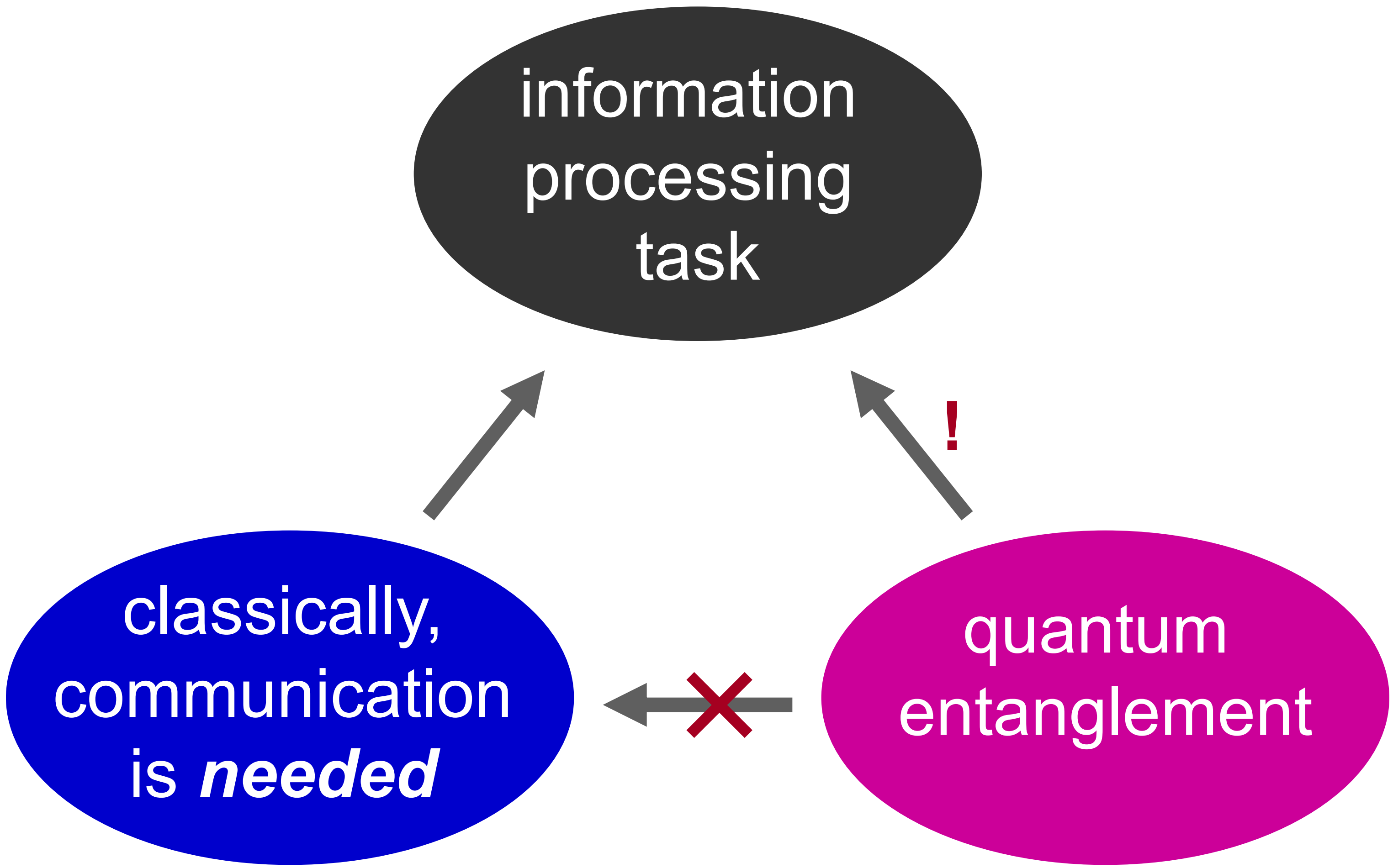


$$\cos(\theta_A + \theta_B)\,(|00\rangle - |11\rangle) + \sin(\theta_A + \theta_B)\,(|01\rangle + |10\rangle)$$

Success probability:

$$\Pr[a \oplus b = s \wedge t] = \cos^2(\pi/8) = \tfrac{1}{2} + \tfrac{1}{4}\sqrt{2} = 0.853\ldots$$

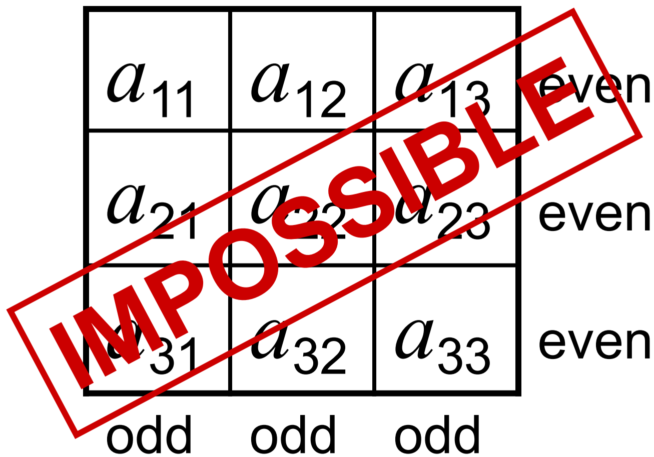
Nonlocality in operational terms

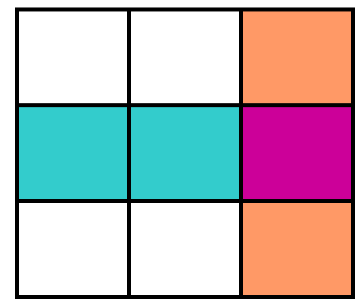


The magic square game

Magic square game

Problem: fill in the matrix with bits such that each row has even parity and each column has odd parity





Game: ask Alice to fill in one row and Bob to fill in one column

They ***win*** iff parities are correct and bits agree at intersection

Success probabilities: $^8/_9$ classical and 1 quantum

[Mermin, 1990] (details omitted here)

Distance measures for quantum states

Distance measures

Some simple (and often useful) measures:

- **Euclidean distance:** $\| \, |\psi\rangle - |\varphi\rangle \, \|_2$
- **Fidelity:** $| \langle\varphi|\psi\rangle |$

Small Euclidean distance implies "closeness" but large Euclidean distance need not (for example, $|\psi\rangle$ vs $-|\psi\rangle$)

Not so clear how to extend these for mixed states …

… though fidelity does generalize, to $\mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$

Trace norm – preliminaries (1)

For a normal matrix M and a function $f: \mathbb{C} \to \mathbb{C}$, we define the matrix $f(M)$ as follows:

$M = U^\dagger D U$, where D is diagonal (i.e. unitarily diagonalizable)

Now, define $f(M) = U^\dagger f(D)\, U$, where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix} \qquad f(D) = \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_d) \end{bmatrix}$$

Trace norm – preliminaries (2)

For a normal matrix $M = U^\dagger D U$, define $|M|$ in terms of replacing D with

$$|D| = \begin{bmatrix} |\lambda_1| & 0 & \cdots & 0 \\ 0 & |\lambda_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\lambda_d| \end{bmatrix}$$

This is the same as defining $|M| = \sqrt{M^\dagger M}$ and the latter definition extends to ***all*** matrices (not necessarily normal ones), since $M^\dagger M$ is positive semidefinite

Trace norm/distance – definition

The ***trace norm*** of M is $||M||_{\mathrm{tr}} = ||M||_1 = \mathrm{Tr}|M| = \mathrm{Tr}\sqrt{M^\dagger M}$

Intuitively, it's the 1-norm of the eigenvalues (or, in the non-normal case, the *singular values*) of M

The ***trace distance*** between ρ and σ is defined as $||\rho - \sigma||_{\mathrm{tr}}$

Why is this a meaningful distance measure between quantum states?

Theorem: for any two quantum states ρ and σ, the ***optimal*** measurement procedure for distinguishing between them succeeds with probability $\frac{1}{2} + \frac{1}{4}||\rho - \sigma||_{\mathrm{tr}}$

Distinguishing between two arbitrary quantum states

Holevo-Helstrom Theorem (1)

Theorem: for any two quantum states ρ and σ, the optimal measurement procedure for distinguishing between them succeeds with probability $\frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_{tr}$ (equal prior probs.)

Proof* (the attainability part):

Since $\rho - \sigma$ is Hermitian, its eigenvalues are real

Let Π_+ be the projector onto the positive eigenspaces

Let Π_- be the projector onto the non-positive eigenspaces

Take the POVM measurement specified by Π_+ and Π_- with the associations $+ \equiv \rho$ and $- \equiv \sigma$

* The other direction of the theorem (optimality) is omitted here

Holevo-Helstrom Theorem (2)

Claim: this succeeds with probability $\frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_{tr}$

Proof of Claim:

A key observation is $\mathrm{Tr}(\Pi_+ - \Pi_-)(\rho - \sigma) = \|\rho - \sigma\|_{tr}$

The success probability is $p_s = \frac{1}{2}\mathrm{Tr}(\Pi_+ \rho) + \frac{1}{2}\mathrm{Tr}(\Pi_- \sigma)$

& the failure probability is $p_f = \frac{1}{2}\mathrm{Tr}(\Pi_+ \sigma) + \frac{1}{2}\mathrm{Tr}(\Pi_- \rho)$

Therefore, $p_s - p_f = \frac{1}{2}\mathrm{Tr}(\Pi_+ - \Pi_-)(\rho - \sigma) = \frac{1}{2}\|\rho - \sigma\|_{tr}$

From this, the result follows ■

Purifications & Ulhmann's Theorem

Any density matrix ρ, can be obtained by tracing out part of some larger ***pure*** state:

$$\rho = \sum_{j=1}^{d} \lambda_j \left|\varphi_j\right\rangle\left\langle\varphi_j\right| = \mathrm{Tr}_2\left(\sum_{j=1}^{m} \sqrt{\lambda_j}\left|\varphi_j\right\rangle\left|j\right\rangle\right)\left(\sum_{j=1}^{m} \sqrt{\lambda_j}\left\langle\varphi_j\right|\left\langle j\right|\right)$$

a purification of ρ

Ulhmann's Theorem*: The ***fidelity*** between ρ and σ is the maximum of $\langle\varphi|\psi\rangle$ taken over all purifications $|\psi\rangle$ and $|\varphi\rangle$

* See [Nielsen & Chuang, pp. 410-411] for a proof of this

Recall our previous definition of fidelity as

$$\mathrm{F}(\rho,\ \sigma) = \mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} \equiv \|\rho^{1/2}\sigma^{1/2}\|_{\mathrm{tr}}$$

Relationships between fidelity and trace distance

$$1 - F(\rho, \sigma) \;\leq\; \|\rho - \sigma\|_{tr} \;\leq\; \sqrt{1 - F(\rho, \sigma)^2}$$

See [Nielsen & Chuang, pp. 415-416] for more details