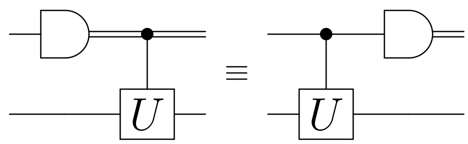
Assignment 3
Due: 11:59pm, October 5, 2021

1. **Measuring the control qubit of a CNOT gate [15 points].** Let U be an arbitrary unitary, and consider these two procedures: (a) measure the control qubit in the computational basis and then perform a classically controlled-U; (b) perform a controlled-U and then measure the control qubit in the computational basis. Show that, for any 2-qubit input state, the result of these two procedures is exactly the same:



In each case, the measurement outcome and residual state can be expressed as

$$\begin{cases} (0, |\psi_0\rangle) & \text{with probability } p_0 \\ (1, |\psi_1\rangle) & \text{with probability } p_1, \end{cases}$$

and you should show that p_0, p_1, $|\psi_0\rangle$, $|\psi_1\rangle$ are the same for both procedures.

2. **Unitary between two pairs of states with the same inner product [15 points].** Let $|\phi_0\rangle$ and $|\phi_1\rangle$ be a pair of *non-orthogonal* 1-qubit states and let $|\psi_0\rangle$ and $|\psi_1\rangle$ be another pair of 1-qubit states with the same inner product. In other words: $\langle\phi_0|\phi_1\rangle = \langle\psi_0|\psi_1\rangle$. Show that there exists a unitary U such that $U|\phi_0\rangle = |\psi_0\rangle$ and $U|\phi_1\rangle = |\psi_1\rangle$.

   (Note: this means that if we have a good distinguishing procedure for $|\psi_0\rangle$ vs. $|\psi_1\rangle$ then, by applying U, it can be used to distinguish equally well for $|\phi_0\rangle$ vs. $|\phi_1\rangle$.)

3. **(This is an optional question for bonus credit)**
   Distinguishing between identical and orthogonal states [8 points]. Consider the problem where you are given two qubits as input and you are promised that *either* they are in the same state *or* they are states that are orthogonal to each other. But you are given no information about what the states actually are. To be clear, when the qubits are the same, the input is of the form $|\phi\rangle|\phi\rangle$, where $|\phi\rangle$ can be *any* 1-qubit state that's unknown to you. And, when the qubits are orthogonal, the input is of the form $|\phi\rangle|\psi\rangle$, where $|\phi\rangle$ and $|\psi\rangle$ can be *any* two 1-qubit states (unknown to you) subject to $\langle\phi|\psi\rangle = 0$.

   Your goal is to determine whether they are the same or orthogonal. Is this at all possible? Although, it's not possible to do this perfectly, there is a procedure that achieves this:

   - Whenever the states are the same, it correctly outputs "same".
   - Whenever the states are orthogonal, it outputs $\begin{cases} \text{“orthogonal”} & \text{with probability } \frac{1}{2} \\ \text{“same”} & \text{with probability } \frac{1}{2}. \end{cases}$

   Give a procedure that achieves this, as a unitary operation followed by a measurement, and explain why it works. You may use an ancilla qubit.