Assignment 9

Due: 11:59pm, Tuesday, November 30, 2021

1. **Correcting errors at known positions [18 points; 6 each].** Here we consider a method of encoding a 2-qubit state as a 4-qubit state that protects against a 1-qubit *erasure* error. An erasure error means that one qubit goes missing, and where we know *which* qubit goes missing. The code is based on these four "logical" states

$$|00\rangle_L = \tfrac{1}{\sqrt{2}}|0000\rangle + \tfrac{1}{\sqrt{2}}|1111\rangle$$
$$|01\rangle_L = \tfrac{1}{\sqrt{2}}|0101\rangle + \tfrac{1}{\sqrt{2}}|1010\rangle$$
$$|10\rangle_L = \tfrac{1}{\sqrt{2}}|1001\rangle + \tfrac{1}{\sqrt{2}}|0110\rangle$$
$$|11\rangle_L = \tfrac{1}{\sqrt{2}}|1100\rangle + \tfrac{1}{\sqrt{2}}|0011\rangle.$$

For any 2-qubit pure state $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$, its encoding is the 4-qubit state $\alpha_{00}|00\rangle_L + \alpha_{01}|01\rangle_L + \alpha_{10}|10\rangle_L + \alpha_{11}|11\rangle_L$.

The claim is that if any one of the four qubits of the encoding goes missing (and we know which qubit is missing) then the 2-qubit data can be recovered from the three remaining qubits.

(a) To begin with, consider the case where the last qubit is missing. Draw a 4-qubit quantum circuit *with gates acting only on the first three qubits* that maps any encoding $\alpha_{00}|00\rangle_L + \alpha_{01}|01\rangle_L + \alpha_{10}|10\rangle_L + \alpha_{11}|11\rangle_L$ to the output state

$$\big(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle\big) \otimes \big(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\big).$$

This can be done with only four CNOT gates (and no other gates); please only use CNOT gates. The existence of such a circuit implies that the 2-qubit data can be recovered from only the first three qubits of the encoding.

(It remains to show that there exist recovery circuits for the other cases of one missing qubit; however, by the symmetries of the encoding, those cases are very similar to the case above, and you are not required to show them.)

(b) Give a circuit that produces the encoding. The input should be a 2-qubit state of the form $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$ and two ancilla qubits in state $|00\rangle$. The output should be the encoding $\alpha_{00}|00\rangle_L + \alpha_{01}|01\rangle_L + \alpha_{10}|10\rangle_L + \alpha_{11}|11\rangle_L$.

(c) So we know that the data can be fully recovered from any three of the four qubits. Suppose that you have access to only (say) the first *two* qubits of the 4-qubit encoding $\alpha_{00}|00\rangle_L + \alpha_{01}|01\rangle_L + \alpha_{10}|10\rangle_L + \alpha_{11}|11\rangle_L$. What is the reduced density matrix of the state of the first two qubits (i.e., with the last two qubits traced out)?

2. **Trace distance between pure states [12 points].** Show that, for states $|0\rangle$ and $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ (where $\theta \in [0, \frac{\pi}{2}]$), the angle between their corresponding points on the Bloch sphere is 2θ.