Assignment 2
Due date: 11:59pm, October 6, 2022

1. **Product states versus entangled states [12 points; 4 for each part].** In each case, either express the state as a tensor product or prove that it cannot be expressed that way:

   (a) $\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$ as a tensor product of two 1-qubit states.

   (b) $\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$ as a tensor product of two 1-qubit states.

   (c) $\frac{1}{2}|000\rangle + \frac{1}{2}|011\rangle + \frac{1}{2}|101\rangle + \frac{1}{2}|110\rangle$ as a tensor product of a 1-qubit state (first qubit) and a 2-qubit state (second and third qubits).

2. **Local operations on Bell states [12 points; 4 each].** Let $U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$ be unitary.

   (a) Show that applying U to the first qubit of the state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is equivalent to applying U^T to the second qubit (where U^T denotes the transpose of U). Namely,

   $$(U \otimes I)\big(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\big) = (I \otimes U^T)\big(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\big).$$

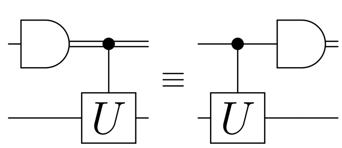
   (b) Give a 2×2 matrix V (with matrix entries in terms of $u_{00}, u_{01}, u_{10}, u_{11}$) such that

   $$(U \otimes I)\big(\tfrac{1}{\sqrt{2}}|01\rangle + \tfrac{1}{\sqrt{2}}|10\rangle\big) = (I \otimes V)\big(\tfrac{1}{\sqrt{2}}|01\rangle + \tfrac{1}{\sqrt{2}}|10\rangle\big).$$

   (c) Give a 2×2 matrix W (with matrix entries in terms of $u_{00}, u_{01}, u_{10}, u_{11}$) such that

   $$(U \otimes I)\big(\tfrac{1}{\sqrt{2}}|00\rangle - \tfrac{1}{\sqrt{2}}|11\rangle\big) = (I \otimes W)\big(\tfrac{1}{\sqrt{2}}|00\rangle - \tfrac{1}{\sqrt{2}}|11\rangle\big).$$

3. **Measuring the control qubit of a CNOT gate [12 points].** Let U be an arbitrary unitary, and consider these two procedures: (a) measure the control qubit in the computational basis and then perform a classically controlled-U; (b) perform a controlled-U and then measure the control qubit in the computational basis. Show that, for any 2-qubit input state, $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$, the result of these two procedures is exactly the same:

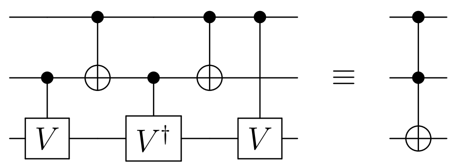


   In each case, the measurement outcome and residual state can be expressed as

   $$\begin{cases} (0,\ |\psi_0\rangle) & \text{with probability } p_0 \\ (1,\ |\psi_1\rangle) & \text{with probability } p_1, \end{cases}$$

   and you should show that p_0, p_1, $|\psi_0\rangle$, $|\psi_1\rangle$ are the same for both procedures.

4. **Constructing a Toffoli gate out of two-qubit gates [12 points].** The Toffoli gate (controlled-controlled-NOT) is a 3-qubit gate, and here we show how to implement it with 2-qubit gates. The construction is given by the following quantum circuit



where

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} \omega & \overline{\omega} \\ \overline{\omega} & \omega \end{bmatrix}, \quad \text{with } \omega = e^{i\pi/4} \text{ and } \overline{\omega} = e^{-i\pi/4} \text{ (ω's conjugate).} \tag{1}$$

We *could* verify this by multiplying 8×8 matrices; however, we take a simpler approach.

(a) [2 points] Show that $V^2 = X$.

(b) [8] Prove each of the following, where $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ is an arbitrary 1-qubit state:

i. The circuit maps $|00\rangle|\psi\rangle$ maps to $|00\rangle|\psi\rangle$.

ii. The circuit maps $|01\rangle|\psi\rangle$ maps to $|01\rangle|\psi\rangle$.

iii. The circuit maps $|10\rangle|\psi\rangle$ maps to $|10\rangle|\psi\rangle$.

iv. The circuit maps $|11\rangle|\psi\rangle$ maps to $|11\rangle V^2|\psi\rangle$.

(c) [2] Based on parts (a) and (b), give the 8×8 unitary matrix of the above circuit.

5. **Distinguishing between pairs of unitaries [12 points; 4 each].** In each case, you are given a black-box gate that computes one of the two given unitaries, but you are not told which one. Your goal is to determine which of the two unitaries it is. To help you do this, you can create any quantum state, apply the black-box gate to this state, and then measure the answer in some basis (that is, you can apply a unitary of your choosing and then measure in the computational basis). You can only use the black-box gate once.

For example, consider the case where the two unitaries are $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In this case, setting the initial state to $|+\rangle$, applying the black-box unitary, followed by H and measuring yields 0 in the first case and 1 in the second case. So this is a perfect distinguishing procedure (it succeeds with probability 1).

Give a perfect distinguishing procedure in each case below.

(a) $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

6. **(This is an optional question for bonus credit)**
Distinguishing among the trine states with two guesses permitted [6 points].
Recall that the three trine states are

$$
\begin{aligned}
|\phi_0\rangle &= |0\rangle \\
|\phi_1\rangle &= -\tfrac{1}{2}|0\rangle + \tfrac{\sqrt{3}}{2}|1\rangle \\
|\phi_1\rangle &= -\tfrac{1}{2}|0\rangle - \tfrac{\sqrt{3}}{2}|1\rangle.
\end{aligned}
$$

Suppose that you are given one of the three trine states (not being told which one) and you can perform any measurement of your choosing. Based on the outcome of the (single) measurement of the (single) state you received, you are permitted *two* guesses of which trine state you received. Give a procedure that maximizes the probability that one of your two guesses is correct.