Assignment 1
Due date: 11:59pm, September 22, 2022

1. **Distinguishing between pairs of states [12 points; 4 for each part].** In each case, one of the two given states is prepared and sent to you (you are not told which one; each case arises with probability $\frac{1}{2}$). Your goal is to guess which of the two states it is. You are allowed to perform any measurement operation on the state to help you with this goal.

   Describe your distinguishing procedure as a unitary operation followed by a measurement (in the computational basis) and give its average-case success probability. (Your assigned grade will depend on how close your average-case success probability is to optimal.)

   (a) $|0\rangle$ vs. $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$

   (b) $\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ vs. $\frac{1}{\sqrt{2}}|0\rangle + \frac{-i}{\sqrt{2}}|1\rangle$ (where $i = \sqrt{-1} = e^{i\pi/2}$)

   (c) $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$ vs. $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$ (in this case you're given two qubits).

2. **Decompositions into rotations and phase rotations [12 points; 4 each].** For all angles $\theta \in [0, 2\pi)$, define $R(\theta)$ and $P(\theta)$ as

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{and} \qquad P(\theta) = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}. \tag{1}$$

   (a) Show that there exist angles $\theta_1, \theta_2 \in [0, 2\pi)$ for which it holds that $iH = P(\theta_1)R(\theta_2)$, where $iH = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} i & i \\ i & -i \end{smallmatrix}\right]$ (and $i = \sqrt{-1}$).

   (b) Define $C = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right]$. Show that there *do not exist* $\theta_1, \theta_2 \in [0, 2\pi)$ for which it holds that $C = P(\theta_1)R(\theta_2)$.

   (c) Show that there exist $\theta_1, \theta_2, \theta_3 \in [0, 2\pi)$ for which it holds that $C = P(\theta_1)R(\theta_2)P(\theta_3)$ (where C is as defined in part (b)).

3. **Applying 1-qubit gates to 2-qubit states [12 points; 6 each].** Consider these two 2-qubit states:

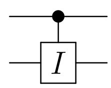
$$\tfrac{1}{\sqrt{2}}|01\rangle + \tfrac{1}{\sqrt{2}}|10\rangle$$
$$\tfrac{1}{\sqrt{2}}|01\rangle - \tfrac{1}{\sqrt{2}}|10\rangle$$

   (a) For each of the two states, calculate the state vector that results if a Hadamard gate H is applied to the *second* qubit (i.e., $I \otimes H$). Recall that $H = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

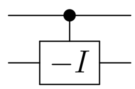
(b) For each of the two states, calculate the state vector that results if a Hadamard gate H is applied to *both* of the qubits (i.e., $H \otimes H$).

4. **The controlled-I and controlled-$(-I)$ gates [12 points].** For parts (a)(b)(c)(d), write out the 4×4 matrix for the 2-qubit unitary operation; then proceed to part (e).

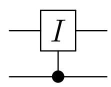
   (a) [2 points]

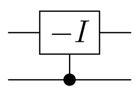
   (b) [2]

   (c) [2]

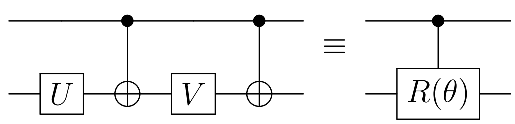
   (d) [2]

   (e) [4] Show that the gate in part (b) can actually be simulated by a single 1-qubit gate.

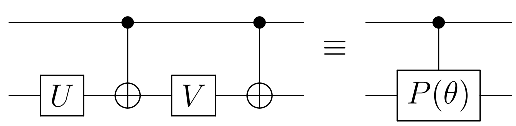
5. **Simulating controlled-$R(\theta)$ and controlled-$P(\theta)$ gates with CNOT and 1-qubit gates [12 points; 6 each].** Let $R(\theta)$ and $P(\theta)$ be as defined in question 2 (Eq. (1)), where where $\theta \in [0, 2\pi)$.

   (a) Show that, for all $\theta \in [0, 2\pi)$, there exist 1-qubit unitaries U and V such that

   (Hint: consider setting U and V to rotation matrices with carefully chosen angles.)

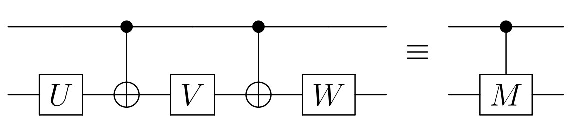
   (b) Show that, for all $\theta \in [0, 2\pi)$, there exist 1-qubit unitaries U and V such that

6. **(This is an optional question for bonus credit) [8 points].**
   Simulating any controlled-$\big(P(\theta_1)R(\theta_2)P(\theta_3)\big)$ gate with CNOT and 1-qubit gates.

   Again, $R(\theta)$ and $P(\theta)$ are as defined in question 2 (Eq. (1)), where $\theta \in [0, 2\pi)$.

   Show that, for all $M = P(\theta_1)R(\theta_2)P(\theta_3)$, where $\theta_1, \theta_2, \theta_3 \in [0, 2\pi)$, there exist 1-qubit unitaries U, V, and W such that

Note: Please read the *Grading policies for assignments* on the course web site for information about how submitted assignments are graded, including the comments about bonus questions.