Assignment 10

Due: 11:59pm, Tuesday, December 7, 2021

1. **A non-local game based on locally mapping trits to bits [20 points].** Consider the following non-local game. Alice and Bob are not allowed to communicate with each other once the game starts. Alice receives $s \in \{0,1,2\}$ as her input and Bob receive $t \in \{0,1,2\}$ as his input (where (s,t) is generated randomly according the uniform distribution on $\{0,1,2\}^2$). They produce output bits $a,b \in \{0,1\}$ (respectively). The winning condition is that: if $s=t$ then $a=b$; and if $s \neq t$ then $a \neq b$.

   (a) [5 points] Show that there is a classical strategy (i.e., without shared entanglement) that wins with probability $7/9 = 0.777\ldots$ for this game.

   (b) [5 points] Prove that no classical strategy (i.e., without shared entanglement) can win with probability higher than 7/9 for this game. In your proof, you may assume that the classical strategy is deterministic (it can be shown that the same bound for probabilistic strategies follows from this).

   (c) [10 points] Give an entangled strategy for this game with the highest winning probability that you can attain. Your credit will depend on how close your strategy's winning probability is to optimal.

   Hint 1: There is an optimal strategy that uses entangled state $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$ and where Alice applies a rotation (where the angle depends on her input s) and measures, and Bob applies a rotation (where the angle depends on his input t) and then measures. Therefore, you may restrict your attention to strategies of this form.

   Hint 2: There are entangled strategies that win with probability higher than 0.9.

2. **The strong correlations of maximally entangled states [10 points].** Suppose that Alice and Bob each have a d-dimensional register which jointly contain the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle \otimes |k\rangle. \tag{1}$$

   Obviously, if they both measure their register in the computational basis then their outcomes will be the same (perfectly correlated). What's remarkable is that there are many *other* orthonormal bases for which this property of perfect correlation also holds.

   Let $|v_0\rangle, |v_1\rangle, \ldots, |v_{d-1}\rangle \in \mathbb{R}^d$ be any real[1] orthonormal basis. Prove that, if Alice and Bob each measure their register with respect to the orthonormal basis $|v_0\rangle, |v_1\rangle, \ldots, |v_{d-1}\rangle$ then their outcomes will be the same.

   In other words, prove that, if they both apply the measurement with POVM elements $|v_0\rangle\langle v_0|$, $|v_1\rangle\langle v_1|$, $\ldots$, $|v_{d-1}\rangle\langle v_{d-1}|$, then Alice's outcome is k if and only if Bob's outcome is k.

   First try to do this without looking at the hint on the next page.

[1] Meaning that all the vectors are in $\mathbb{R}^d$ (as opposed to $\mathbb{C}^d$).

Hint: First prove that $(M \otimes I)|\psi\rangle = (I \otimes M^T)|\psi\rangle$ (where M^T is the transpose of M).