Assignment 1
Due date: 11:59pm, September 26, 2023

1. **Distinguishing between pairs of states [12 points; 3 for each part].** In each case, one of the two given states is prepared and sent to you (you are not told which one; each case arises with probability $\frac{1}{2}$). Your goal is to guess which of the two states it is. You are allowed to perform any measurement operation on the state to help you with this goal.

   Describe your distinguishing procedure as a unitary operation followed by a measurement (in the computational basis) and give its average-case success probability. (Your assigned grade will depend on how close your average-case success probability is to optimal.)

   (a) $|0\rangle$ vs. $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$

   (b) $\frac{3}{5}|0\rangle + \frac{4}{5}i|1\rangle$ vs. $\frac{4}{5}i|0\rangle + \frac{3}{5}|1\rangle$ (where $i = \sqrt{-1} = e^{i\pi/2}$)

   (c) $|0\rangle|0\rangle$ vs. $|+\rangle|+\rangle$ (where $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$)

   (d) $|0\rangle|0\rangle$ vs. $\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$

2. **Simple operations on quantum states [12 points; 2 for each part].** In each case, describe the resulting state (where H is the 2×2 Hadamard transform).

   (a) Apply H to the qubit in state $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.

   (b) Apply H to first qubit of state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

   (c) Apply H to both qubits of state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

   (d) Apply $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ to both qubits of state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. ($i = \sqrt{-1}$)

   (e) Apply H to all three qubits of state $\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$.

   (f) Apply H to first qubit of state $\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$, and then measure this first qubit (in the computational basis). Here, you should give the state of the two remaining qubits in each of two cases:

      i. when the outcome of the measurement is 0;

      ii. when the outcome of the measurement is 1.

3. **Entangled states and product states [12 points; 4 for each part].** For each two-qubit state below, either express it as a product of two one-qubit states or show that such a factorization is impossible (in the latter case, the qubits are *entangled*).

   (a) $\frac{1}{2}|00\rangle + \frac{1}{2}i|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}i|11\rangle$

   (b) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

   (c) $0.36|00\rangle + 0.48|01\rangle + 0.48|10\rangle + 0.64|11\rangle$

4. **Simple quantum circuit constructions [12 points; 4 each].**

(a) Describe a two-qubit quantum circuit consisting of *one* CNOT gate and *two* H (Hadamard) gates that computes the following unitary tranformation:

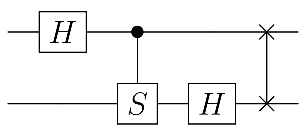
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(b) Define the one-qubit gates H and S as

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad (\text{where } i = \sqrt{-1}).$$

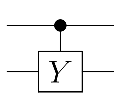
In each case, give the 4×4 unitary matrix corresponding to the two-qubit gate:

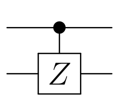
(c) Give the 4×4 unitary matrix corresponding to the following quantum circuit

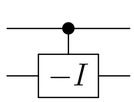


where S is as defined in part (b), and the last (two-qubit) gate denotes a *swap gate*, that transposes the two qubits (more precisely, a swap gate maps $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |10\rangle$, $|10\rangle \mapsto |01\rangle$, and $|11\rangle \mapsto |11\rangle$).

5. **Gates that map some product state to an entangled state [12 points; 4 each].** Suppose that we start with the product state $|+\rangle|0\rangle$, and apply a controlled-X (a.k.a. CNOT) gate to it. The resulting state is $\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$, which is of course entangled. Call a 2-qubit gate *entangling* if there exists at least one product state that the gate maps to an entangled state. In each case, state whether the gate is entangling or not and prove that your answer is correct.

(a) 

(b) 

(c) 

6. **(This is an optional question for bonus credit)**
Distinguishing among three qutrit states [8 points].
Here, the states are qutrits, where the computational basis states are $|0\rangle$, $|1\rangle$, and $|2\rangle$. Let $|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, $|\psi_1\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle$, and $|\psi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle + \frac{1}{\sqrt{2}}|0\rangle$. Note that these states are not orthogonal to each other and therefore not perfectly distinguishable. Describe a procedure that distinguishes among these states with as high average-case success probability as possible (where the input state is $|\psi_k\rangle$, for a randomly chosen $k \in \{0, 1, 2\}$, according to the uniform distribution). Describe the procedure, state its success probability, and explain why it works.

Note: Please read the *Grading policies for assignments* on the course web site for information about how submitted assignments are graded, including the comments about bonus questions.