Assignment 7

Due date: 11:59pm, November 5, 2020

1. **Efficiently computing bijections [6 points].** Assume that $f\{0,1\}^n \to \{0,1\}^n$ is a bijection. In that case, there exists an n-qubit unitary operation U_f such that $U_f|a\rangle = |f(a)\rangle$ for all $a \in \{0,1\}^n$. Here we consider conditions under which U_f can be computed efficiently in terms of 1- and 2-qubit gates.

   Assume that we have an efficient classical algorithm that computes $f(a)$ from a. From material in the lectures, we can deduce from this that an f-query can be efficiently computed. Denote this operation by Q_f, where $Q_f|a\rangle|b\rangle = |a\rangle|f(a) \oplus b\rangle$.

   Assume that we also have an efficient classical algorithm that computes $f^{-1}(a)$ from a. From material in the lectures, we can deduce from this that $Q_{f^{-1}}|a\rangle|b\rangle = |a\rangle|f^{-1}(a) \oplus b\rangle$ can be efficiently computed.

   Show how to combine Q_f and $Q_{f^{-1}}$ and possibly $O(n)$ additional 2-qubit gates to compute U_f. You may use an n-qubit ancilla (thereby mapping $|a\rangle|00\ldots0\rangle$ to $|f(a)\rangle|00\ldots0\rangle$).

2. **Basic questions about density matrices [12 points; 4 each].**

   (a) Show that, for any operator that is Hermitian, positive definite (i.e., has no negative eigenvalues), and has trace 1, there is a probabilistic mixture of pure states whose density matrix is ρ.

   (b) A density matrix ρ corresponds to a *pure* state if and only if $\rho = |\psi\rangle\langle\psi|$. Show that any density matrix ρ corresponds to a pure state if and only if $\mathrm{Tr}(\rho^2) = 1$.

   (c) Show that every 2×2 density matrix ρ can be expressed as an *equally weighted mixture* of pure states. That is

   $$\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$$

   for states $|\psi_1\rangle$ and $|\psi_2\rangle$. (Hint: $|\psi_1\rangle$ and $|\psi_2\rangle$ don't have to be orthogonal.)

3. **The density matrix depends on what you know [12 points; 3 each].** Consider the following scenario. Alice first flips a fair coin that has outcome 0 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$. If the coin value is 0 she creates the state $|0\rangle$ and if the coin value is 1 she creates the state $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$. Then Alice sends the state that she created to Bob (she does not send the coin value).

   (a) From Alice's perspective (who *knows* the coin value), the density matrix of the state she created will be either $|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $|+\rangle\langle +| = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. What is the density matrix of the state from Bob's perspective (who does *not* know the coin value)? Give the four matrix entries of this density matrix.

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(b) Suppose that, upon receiving the state from Alice, Bob measures it in the computational basis. The measurement process yields a classical bit and an output state ("collapsed" to $|0\rangle$ or $|1\rangle$).

Alice knows which state she prepared at the beginning and she knows what measurement Bob applied to the state, but Alice does not know the outcome of Bob's measurement. What's Alice's density matrix for Bob's state assuming that her initial state was $|0\rangle$? What's Alice's density matrix for Bob's state assuming that her initial state was $|+\rangle$?

Suppose that we modify the above scenario to one where Alice flips a *biased* coin whose outcome is 0 with probability $\cos^2(\pi/8)$ and 1 with probability $\sin^2(\pi/8)$. If the coin value is 0 she creates the state $|\psi_0\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$ and if the coin value is 1 she creates the state $|\psi_1\rangle = \sin(\pi/8)|0\rangle - \cos(\pi/8)|1\rangle$. Alice sends the state (but not the coin value) to Bob.

(c) From Alice's perspective (who *knows* the coin value), the density matrix of the state she created will be either $|\psi_0\rangle\langle\psi_0|$ or $|\psi_1\rangle\langle\psi_1|$. What is the density matrix of the state from Bob's perspective (who does *not* know the coin value)? Give the four matrix entries of this density matrix. Is it the same matrix as in part (a)?

(d) Suppose that, upon receiving the state from Alice, Bob measures it in the computational basis. The measurement process yields a classical bit and an output state ("collapsed" to $|0\rangle$ or $|1\rangle$).

Alice knows which state she prepared at the beginning and she knows what measurement Bob applied to the state, but Alice does not know the outcome of Bob's measurement. What's Alice's density matrix for Bob's state assuming that her initial state was $|\psi_0\rangle$? What's Alice's density matrix for Bob's state assuming that her initial state was $|\psi_1\rangle$?