Assignment 6

Due date: 11:59pm, **October 30**, 2020

1. **Quantum Fourier transform [20 points].** Let F_m denote the m-dimensional Fourier transform

$$F_m = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{m-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{m-1} & \omega^{2(m-1)} & \cdots & \omega^{(m-1)^2} \end{bmatrix}, \quad \text{where } \omega = e^{2\pi i/m} \ (i = \sqrt{-1}).$$

   (a) [10 points] Prove that F_m is unitary.

   (b) [10 points] Whenever $m > 2$, $(F_m)^2 \neq I$; however, $(F_m)^2$ maps computational basis states to computational basis states. For $a \in \mathbb{Z}_m$, what computational basis state is $(F_m)^2|a\rangle$? Give a simple expression and justify your answer.

2. **Superposition of eigenvectors in phase estimation algorithm [10 points].** This question is about what happens when the quantum algorithm for phase-estimation is applied to a superposition of two eigenvectors. Let U be an n-qubit unitary and let W be a multiplicity-controlled-U gate with ℓ control qubits. That is, for all $x \in \{0,1\}^\ell$, $y \in \{0,1\}^n$, $W|x\rangle|y\rangle = |x\rangle U^x|y\rangle$.

   Let $|\psi_1\rangle$, $|\psi_2\rangle$ be two orthogonal eigenvectors of U, with respective eigenvalues $e^{2\pi i\phi_1}$ and $e^{2\pi i\phi_2}$, where $\phi_1 = a/2^\ell$ and $\phi_2 = b/2^\ell$ ($a, b \in \mathbb{Z}_{2^\ell} \equiv \{0,1\}^\ell$).

   We know from the exact case of the Phase-Estimation Algorithm that

$$(F_{2^\ell}^* \otimes I)W(H^{\otimes \ell} \otimes I)|0^\ell\rangle|\psi_1\rangle = |a\rangle|\psi_1\rangle \tag{1}$$

$$(F_{2^\ell}^* \otimes I)W(H^{\otimes \ell} \otimes I)|0^\ell\rangle|\psi_2\rangle = |b\rangle|\psi_2\rangle. \tag{2}$$

   Let $\alpha_1, \alpha_2 \in \mathbb{C}$ be such that $|\alpha_1|^2 + |\alpha_2|^2 = 1$. Describe the result of a measurement in the computational basis of the first ℓ qubits of the state

$$(F_{2^\ell}^* \otimes I)W(H^{\otimes \ell} \otimes I)|0^\ell\rangle(\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle). \tag{3}$$

   (Hint: The answer and justification are simple.)