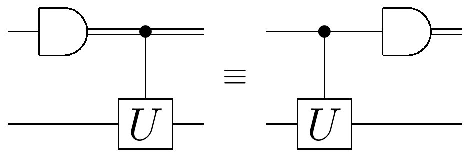
Assignment 3
Due date: 11:59pm, October 1, 2020

1. **Measuring the control qubit of a CNOT gate [15 points].** In Lecture 2, we saw that controlled-U gates can behave in subtle ways, sometimes even changing the state of their control qubits. Here, we consider another aspect of controlled-U gates.

   Let U be an arbitrary unitary, and consider these two procedures: (a) measure the control qubit in the computational basis and then perform a controlled-U; (b) perform a controlled-U and then measure the control qubit in the computational basis. Show that, for any 2-qubit state, the result of these two procedures is the same. More precisely, that both procedures result in the same probability distribution on quantum states for the target qubit.



2. **Unitary between two triples of states with same inner products [15 points].** Let $|\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle \in \mathbb{C}^3$ be a triple of qutrit states and let $|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^3$ be another triple of qutrit states. The triples are not assumed to be orthogonal, but suppose that all the pair-wise inner products in the first triple ($\langle\phi_0|\phi_1\rangle$, $\langle\phi_0|\phi_2\rangle$, and $\langle\phi_1|\phi_2\rangle$) match all the pair-wise inner products in the second triple ($\langle\psi_0|\psi_1\rangle$, $\langle\psi_0|\psi_2\rangle$, and $\langle\psi_1|\psi_2\rangle$). In other words, we have $\langle\phi_0|\phi_1\rangle = \langle\psi_0|\psi_1\rangle$, $\langle\phi_0|\phi_2\rangle = \langle\psi_0|\psi_2\rangle$, and $\langle\phi_1|\phi_2\rangle = \langle\psi_1|\psi_2\rangle$.

   Show that there exists a unitary U such that $U|\phi_0\rangle = |\psi_0\rangle$, $U|\phi_1\rangle = |\psi_1\rangle$, $U|\phi_2\rangle = |\psi_2\rangle$.

   (**Hint:** you might start with an easier problem, where there are only *pairs* of states $|\phi_0\rangle, |\phi_1\rangle \in \mathbb{C}^3$ and $|\psi_0\rangle, |\psi_1\rangle \in \mathbb{C}^3$, and where $\langle\phi_0|\phi_1\rangle = \langle\psi_0|\psi_1\rangle$. You will receive up to 10 points for this question if you show that, in this case, there exists a unitary U such that $U|\phi_0\rangle = |\psi_0\rangle$ and $U|\phi_1\rangle = |\psi_1\rangle$.)

3. **(This is an optional question for bonus credit)**
   A variation of the guess-the-state game [8 points]. In class, we've seen a few games where you receive a quantum system whose state is from some given set of states and your goal is to correctly determine which state it is by some measurement procedure. Here we consider a *variation* of this game—where the goal is to always guess wrongly!

   Let $|\psi_0\rangle, |\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^3$ be three qutrit states whose pairwise inner products are all $\frac{12}{25}$. That is, $\langle\psi_0|\psi_1\rangle = \langle\psi_0|\psi_2\rangle = \langle\psi_1|\psi_2\rangle = \frac{12}{25}$.

   Suppose that you're given a qutrit in one of these three states, but not told which one. Your goal is to *make a guess that is always wrong.* In other words:

   - If the qutrit is $|\psi_0\rangle$ then the only allowable guesses are 1 and 2.
   - If the qutrit is $|\psi_1\rangle$ then the only allowable guesses are 0 and 2.
   - If the qutrit is $|\psi_2\rangle$ then the only allowable guesses are 0 and 1.

   Give a measurement procedure that accomplishes this perfectly. (For this question, you may use the result in question 2.)