Assignment 10

Due date: 11:59pm, December 7 (Monday), 2020

1. **Two reflections is a rotation [12 points].**

   For $\theta \in [0, \pi]$, consider the orthonormal basis

   $$|\psi_\theta\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle \qquad \text{and} \qquad |\psi_\theta^\perp\rangle = -\sin\theta|0\rangle + \cos\theta|1\rangle. \tag{1}$$

   Define the reflections $R_1 = |\psi_{\theta_1}\rangle\langle\psi_{\theta_1}| - |\psi_{\theta_1}^\perp\rangle\langle\psi_{\theta_1}^\perp|$ and $R_2 = |\psi_{\theta_2}\rangle\langle\psi_{\theta_2}| - |\psi_{\theta_2}^\perp\rangle\langle\psi_{\theta_2}^\perp|$. Prove that R_1R_2 is a rotation by angle $2(\theta_1 - \theta_2)$.

2. **Searching when the fraction of marked items is $1/4$ [18 points].**

   Suppose that $f : \{0,1\}^n \to \{0,1\}$ has the property that, for exactly $\frac{1}{4}2^n$ of the values of $x \in \{0,1\}^n$, $f(x) = 1$. Let the goal be to find such an $x \in \{0,1\}^n$ such $f(x) = 1$. Note that there's a simple classical algorithm that finds such an x with high probability with few queries (because a random query succeeds with probability $1/4$). What if we want to solve this problem *exactly* (i.e., with error probability 0)?

   (a) [4] Show that, for any classical algorithm, the number of f-queries required to solve this problem exactly is exponential in n.

   (b) [14] Show that there is a quantum algorithm that makes one single f-query and is guaranteed to find an $x \in \{0,1\}^n$ such $f(x) = 1$. (Hint: consider what a single iteration of Grover's algorithm does.)

3. **(This is an optional question for bonus credit)**
   Searching when the fraction of marked items is $1/2$? [8 points].

   This is the same as part 2(b), but with the assumption that f has the property that, for exactly $\frac{1}{2}2^n$ of the values of $x \in \{0,1\}^n$, $f(x) = 1$. Can the x still be found exactly with one f-query? Either give a quantum algorithm that solves this problem with a single f-query or prove that none exists.