QIC710/CS768/CO681/PH767/AM871 Introduction to Quantum Information Processing (F19)

Assignment 1
Due date: September 19, 2019

1. **Simple operations on quantum states [12 points; 3 for each part].** Let $\theta \in [0, 2\pi]$. In each case, describe the resulting state, where R_θ is the 2×2 rotation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1}$$

   (a) Apply R_θ to the qubit in state $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.

   (b) Apply R_θ to the *first* qubit of state $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$.

   (c) Apply R_θ to *both* qubits of state $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$.

   (d) Apply $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ to *both* qubits of state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. $\quad (i = \sqrt{-1})$

2. **Distinguishing between pairs of quantum states [12 points; 4 each].** In each case, one of the two given states is randomly selected (probability 1/2 each) and given to you. You are not told which one it is. Your goal is to guess which state was selected with as high a probability as you can achieve. Describe your distinguishing procedure as a unitary operation followed by a measurement (in the computational basis) and give its success probability. (Your assigned grade will depend on how close your distinguishing procedure is to optimal.)

   (a) $|0\rangle$ and $|+\rangle$ (recall that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$).

   (b) $|0\rangle|0\rangle$ and $|+\rangle|+\rangle$ (two copies of the state in part (a))

   (c) $|0\rangle|0\rangle$ and $\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$

3. **Distinguishing between two sets of quantum states [12 points, 6 each].** Let $A = \{|0\rangle, |-\rangle\}$ and $B = \{|1\rangle, |+\rangle\}$ (recall that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$).

   (a) Suppose that you are given one of the four states $|0\rangle$, $|+\rangle$, $|1\rangle$, $|-\rangle$ as input and your goal is to determine whether it is in set A or in set B (but not necessarily the state itself). Describe a procedure that, for any of the four possible input states, succeeds at this task with probability $\cos^2(\pi/8)$.

   (b) Same question, but with $A = \{|0\rangle, |+\rangle\}$ and $B = \{|1\rangle, |-\rangle\}$.

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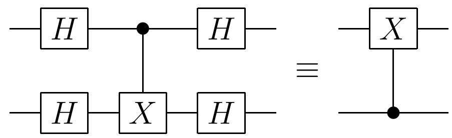
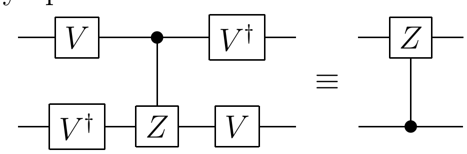
4. **Control-target inversion [12 points; 6 each].** Recall that



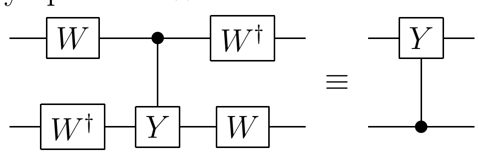
where H is the Hadamard gate and the controlled-X gate is a CNOT gate. Here we consider an analogous property for the other Pauli matrices, namely the controlled-Z and controlled-Y gates, where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2}$$

(a) Give a 2×2 unitary operation V such that



(b) Give a 2×2 unitary operation W such that



Hint: the unitary V for case (a) is extremely simple; the unitary W for case (b) is somewhat more challenging to find.

5. **Product states versus entangled states [12 points; 4 for each part].** In each case, either express the 2-qubit state as a tensor product of 1-qubit states or prove that it cannot be expressed this way:

(a) $\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$

(b) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

(c) $\frac{3}{4}|00\rangle + \frac{\sqrt{3}}{4}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle$

6. **Optional question for bonus credit: distinguishing among three qutrtit states [10 points].**
Here, the states are qutrits, where the computational basis states are $|0\rangle$, $|1\rangle$, and $|2\rangle$. Let $|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$, $|\psi_1\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle$, and $|\psi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle + \frac{1}{\sqrt{2}}|0\rangle$. Note that these states are not orthogonal to each other and therefore not perfectly distinguishable. Describe a procedure that distinguishes among these states with as high a probability as possible (i.e, the input state is $|\psi_k\rangle$, for some (unknown) randomly chosen $k \in \{0, 1, 2\}$ and the goal is to determine k).