QIC710/CS768/CO681/PH767/AM871 Introduction to Quantum Information Processing (F16)

Assignment 4
Due date: November 17, 2016

1. **Haramard transform on uniform superposition of affine linear space [12 points; 6 each].** Let C be any linear subspace of $\{0,1\}^n$ (viewed as a vector space over $\mathbb{Z}_2$). Define $C^\perp = \{x \in \{0,1\}^n :$ such that $x \cdot y = 0$ for all $y \in C\}$, where we define $x \cdot y = x_1 y_1 + \cdots + x_n y_n \bmod 2$.

   (a) Prove that $H^{\otimes n}\left(\frac{1}{\sqrt{|C|}} \sum_{x \in C} |x\rangle\right) = \frac{1}{\sqrt{|C^\perp|}} \sum_{y \in C^\perp} |y\rangle$.

   (b) Prove that, for any $z \in \{0,1\}^n$, $H^{\otimes n}\left(\frac{1}{\sqrt{|C|}} \sum_{x \in C} |x+z\rangle\right) = \frac{1}{\sqrt{|C^\perp|}} \sum_{y \in C^\perp} (-1)^{y \cdot z} |y\rangle$.

   Note: Since part (b) subsumes part (a), a correct solution to (b) alone results in full marks for (a). The purpose of part (a) is as a warm-up to part (b).

2. **An error-correcting encoding of a qubit into three qutrits [12 points].** Consider the following encoding of a qubit into three qu*trits* (where the computational basis states for a qutrit are $|0\rangle$, $|1\rangle$, $|2\rangle$). Each qubit state of the form $\alpha|0\rangle + \beta|1\rangle$ is mapped to the three qutrit *encoded state*

$$\frac{\alpha}{\sqrt{3}}\big(|000\rangle + |111\rangle + |222\rangle\big) + \frac{\beta}{\sqrt{6}}\big(|012\rangle + |021\rangle + |102\rangle + |120\rangle + |201\rangle + |210\rangle\big). \quad (1)$$

   We will show that this code can handle the following type of calamity: one of the three qutrits goes missing (where we do not know which one) and the remaining two qutrits are given to us in an arbitrary order. So what we're left with is two qutrits that are qutrit i and qutrit j for *some* $i, j \in \{0,1,2\}$, but we have no idea what i and j are.

   (a) [3 points] Consider the linear operator M on two-qutrit states such that, for all $a, b \in \{0,1,2\}$, $M|a,b\rangle = |2a + b \bmod 3, a + b \bmod 3\rangle$. Show that this M is unitary.

   (b) [3 points] Show that, if M is applied to the first two qutrits of the above encoded state then it is transformed to

$$\big(\alpha|0\rangle + \tfrac{1}{\sqrt{2}}\beta(|1\rangle + |2\rangle)\big) \otimes \tfrac{1}{\sqrt{3}}(|00\rangle + |12\rangle + |21\rangle). \quad (2)$$

   (c) [3 points] Assume the results in part (a) and (b) are true, and show how to recover the qubit from the state of the first two qutrits of the encoded state.

   (d) [3 points] Assume a solution to part (c) and show how to recover the qubit from the state of any two of the qutrits in a manner that does not require us to know which two qutrits they are (or in what order they are given). (Hint: symmetry)

   (e) [4 points] Now, suppose that you are in possession of only the first qutrit. Prove that absolutely no information about the original qubit can be deduced from this.

3. **General conversion from Stinespring form to Krauss form [12 points].** Suppose that you are given a description of a quantum operation that takes an n-qubit state ρ as input and produces an n'-qubit state σ as output, where the description is of the following form (where $n + m = n' + m'$):

   i. Append an m qubits, in state $|0^m\rangle$ to the end of the input state.
   ii. Apply an $(n+m)$-qubit unitary operation U.
   iii. Trace out the *first* m' qubits (resulting in an n'-qubit output).

   Show how to implement this in Krauss form as

$$\rho \mapsto \sum_{j \in S} A_j \rho A_j^\dagger,$$

   where $\sum_{j \in S} A_j^\dagger A_j = I$. Please be careful with the dimensions of your matrices/vectors (so that they make sense). Also, to avoid ambiguity between multiplication and tensor product, write $\otimes$ explicitly to denote the latter (it will be assumed that AB means the matrix product of A and B, as opposed to $A \otimes B$).

4. **Trace distance between pure states [12 points; 6 each].**

   (a) Calculate an expression for the trace distance between $|0\rangle$ and $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ as a function of θ.

   (b) Calculate an expression for the Euclidean distance between the two points in the Bloch sphere that correspond to the pure states $|0\rangle$ and $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$.

5. **Constructing an OR gate as a quantum operation [12 points; 6 each].** Recall the binary OR operation, denoted as $\vee$, defined as

a	b	$a \vee b$
0	0	0
0	1	1
1	0	1
1	1	1

   Here we consider operations that map the two-qubit state $|a, b\rangle$ to the one-qubit state $|a \vee b\rangle$, for all $a, b \in \{0, 1\}$. Of course, no unitary operation can perform this mapping, since the input and output dimension do not match; however, general quantum operations can compute this mapping.

   (a) Give a sequence of 2×4 matrices $A_1, \ldots, A_k$ such that $\sum_{j=1}^{k} A_j^\dagger A_j = I$ that compute the OR operation in the sense that, for all $a, b \in \{0, 1\}$, when $\rho = |a, b\rangle\langle a, b|$,

$$\sum_{j=1}^{k} A_j \rho A_j^\dagger = |a \vee b\rangle\langle a \vee b|.$$

   (b) Your operation from part (a) maps all basis states to pure states. Does it map all pure input states to pure output states? Either prove the answer is yes, or provide a counterexample.