QIC710/CS768/CO681/PH767/AM871 Introduction to Quantum Information Processing (F16)

Assignment 1
Due date: September 27, 2016

1. **Simple operations on quantum states [12 points; 2 for each part].** In each case, describe the resulting state. (H is the 2×2 Hadamard transform: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$)

   (a) Apply H to the qubit in state $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$.

   (b) Apply H to first qubit of state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

   (c) Apply H to both qubits of state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

   (d) Apply $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ to both qubits of state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. $\quad (i = \sqrt{-1})$

   (e) Apply H to all three qubits of state $\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$.

   (f) Apply H to first qubit of state $\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$, and then measure this first qubit (in the computational basis). Here, you should give the state of the two remaining qubits in each of two cases:

      i. when the outcome of the measurement is 0;

      ii. when the outcome of the measurement is 1.

2. **Distinguishing between pairs of quantum states [12 points; 4 for each part].** In each case, one of the two given states is randomly selected (probability 1/2 each) and given to you. You are not told which one it is. Your goal is to guess which state was selected with as high a probability as you can achieve. Describe your distinguishing procedure as a unitary operation followed by a measurement (in the computational basis) and give its success probability. (Your assigned grade will depend on how close your distinguishing procedure is to optimal.)

   (a) $|0\rangle$ and $|+\rangle$ $\quad$ (recall that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$).

   (b) $|0\rangle|0\rangle$ and $|+\rangle|+\rangle$ $\quad$ (two copies of the state in part (a))

   (c) $|0\rangle|0\rangle$ and $\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$

3. **Product states versus entangled states [12 points; 4 each].** In each case, either express the 2-qubit state as a tensor product of 1-qubit states or prove that it cannot be expressed this way:

   (a) $\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$

   (b) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

   (c) $\frac{3}{4}|00\rangle + \frac{\sqrt{3}}{4}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle$

4. **Distinguishing states by local measurements [12 points; 4 each].** In this question, we suppose Alice and Bob (who are physically separated from each other, say, in separate labs) are each given one of the qubits of some 2-qubit state. Working as a team, they are required to distinguish between State I and State II with only *local* measurements. We will take this to mean that they can each perform a local (one-qubit) unitary operation and then a measurement (in the computational basis) of their own qubit. After their measurements, they can send only *classical* bits to each other.

   In each case below, either give a perfect distinguishing procedure (that never errs) or explain why there is no perfect distinguishing procedure (i.e., that for any procedure the success probability must be less than 1).

   (a) State I: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
   State II: $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

   (b) State I: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
   State II: $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

   (c) State I: $\frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)$ $(i = \sqrt{-1})$
   State II: $\frac{1}{\sqrt{2}}(|00\rangle - i|11\rangle)$

5. ***Optional* challenge question for bonus credit [10 points].** Question 4, but where Alice and Bob each receive a qu*trit* and the two qutrits are in one of these two states:
   State I: $\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle + \frac{1}{\sqrt{2}}|22\rangle$
   State II: $\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle - \frac{1}{\sqrt{2}}|22\rangle$
   (Note: This is intended to be very challenging.)

6. **Teleporting part of an entangled state [12 points].**

   Recall that, in the teleportation protocol, Alice and Bob initially have a joint state of the form

$$\big(\alpha|0\rangle_{\mathrm{A}} + \beta|1\rangle_{\mathrm{A}}\big)\big(\tfrac{1}{\sqrt{2}}|00\rangle_{\mathrm{AB}} + \tfrac{1}{\sqrt{2}}|11\rangle_{\mathrm{AB}}\big)$$

   (where the subscripts are to emphasize who possesses each qubit). At the end of the protocol, there remains only Bob's qubit, and it is in state $\alpha|0\rangle_{\mathrm{B}} + \beta|1\rangle_{\mathrm{B}}$.

   Suppose that we introduce a third party, Carol, and that Alice's qubit-to-be-teleported is entangled with Carol's qubit, in state $\frac{1}{\sqrt{2}}|00\rangle_{\mathrm{CA}} + \frac{1}{\sqrt{2}}|11\rangle_{\mathrm{CA}}$. Set the initial state to

$$\big(\tfrac{1}{\sqrt{2}}|00\rangle_{\mathrm{CA}} + \tfrac{1}{\sqrt{2}}|11\rangle_{\mathrm{CA}}\big)\big(\tfrac{1}{\sqrt{2}}|00\rangle_{\mathrm{AB}} + \tfrac{1}{\sqrt{2}}|11\rangle_{\mathrm{AB}}\big)$$

   and perform the teleportation protocol on Alice's and Bob's qubits. At the end of the protocol, there will remain two qubits: Carol's and Bob's. Will the joint state of Carol and Bob's qubits be in state

$$\tfrac{1}{\sqrt{2}}|00\rangle_{\mathrm{CB}} + \tfrac{1}{\sqrt{2}}|11\rangle_{\mathrm{CB}}\,?$$

   Justify your answer by giving a *clear* proof that your answer is correct.