QIC710/CS768/CO681/PH767/AM871 Introduction to Quantum Information Processing (F15)

Assignment 1
Due date: September 29, 2015

1. **Distinguishing between pairs of quantum states [20 points; 5 for each part].** In each case, one of the two given states is randomly selected (probability 1/2 each) and given to you. You are not told which one it is. Your goal is to guess which state was selected with as high a probability as you can achieve. Describe your distinguishing procedure as a unitary operation followed by a measurement (in the computational basis) and give its success probability. (Your assigned grade will depend on how close your distinguishing procedure is to optimal.)

   (a) $|0\rangle$ and $|+\rangle$ (recall that $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$).

   (b) $|0\rangle|0\rangle$ and $|+\rangle|+\rangle$ (note that this is like part (a), except you provided with two copies of the state)

   (c) $|0\rangle|0\rangle$ and $\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$

   (d) $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle + e^{-i\theta}|1\rangle)$ (where $\theta \in [0, \pi/2]$ is known to you and your answer should be a function of θ)

2. **Distinguishing between identical and orthogonal states [10 points: 5 each].** Here we consider the problem where you are given two qubits as input that are either in the same state or their states are orthogonal to each other and the goal is to determine whether they are the same or not. We consider processes of this form: apply some 2-qubit unitary operation U to the 2-qubit system and then measure the first qubit in the computational basis. If the two states are identical then outcome should be 0. If the two states are orthogonal then the outcome should be 1.

   (a) Describe a unitary U that perfectly distinguishes (in the above sense) in the special case where the input qubits are in computational basis states. That is, the outcome should be 0 for $|0\rangle|0\rangle$ and $|1\rangle|1\rangle$ and the outcome should be 1 for $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$.

   (b) Prove that, for *all* unitary operations U that perfectly distinguish for computational basis states (such as the one in part (a)), they cannot also perfectly distinguish for states of the form $|+\rangle|+\rangle$, $|+\rangle|-\rangle$, $|-\rangle|+\rangle$, and $|-\rangle|-\rangle$. What is the maximum success probability possible for these states? (Recall that $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$.)

3. **Distinguishing states by local measurements [15 points; 5 each].** In this question, we suppose Alice and Bob (who are physically separated from each other, say, in separate labs) are each given one of the qubits of some 2-qubit state. Working as a team, they are required to distinguish between State I and State II with only *local* measurements. We will take this to mean that they can each perform a local (one-qubit) unitary operation and then a measurement (in the computational basis) of their own qubit. After their measurements, they can send only *classical* bits to each other.

   In each case below, either give a perfect distinguishing procedure (that never errs) or explain why there is no perfect distinguishing procedure (i.e., that for any procedure the success probability must be less than 1).

(a) State I: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
State II: $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

(b) State I: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
State II: $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

(c) State I: $\frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)$ $\quad (i = \sqrt{-1})$
State II: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

4. **Optional challenge question for bonus credit [10 points].** Question 3, but where Alice and Bob each receive a qu*trit* and the two qutrits are in one of these two states:
State I: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}\big(\frac{1}{\sqrt{2}}|11\rangle + \frac{1}{\sqrt{2}}|22\rangle\big)$
State II: $\frac{1}{\sqrt{2}}|00\rangle$

5. **Simple quantum circuit constructions [15 points; 5 each].**

(a) Describe a two-qubit quantum circuit consisting of *one* CNOT gate and *two* Hadamard gates that computes the following unitary tranformation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(b) Define the one-qubit gates H and S as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (\text{where } i = \sqrt{-1}).$$

In each case, give the 4×4 unitary matrix corresponding to the two-qubit controlled gate:

$$\begin{array}{cccc} \bullet & H & \bullet & S \\ H & \bullet & S & \bullet \end{array}$$

(c) Give the 4×4 unitary matrix corresponding to the following quantum circuit

$$\begin{array}{cccc} H & \bullet & & \times \\ & S & H & \times \end{array}$$

where S is as defined in part (b), and the last (two-qubit) gate denotes a *swap gate*, that transposes the two qubits (more precisely, a swap gate maps $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |10\rangle$, $|10\rangle \mapsto |01\rangle$, and $|11\rangle \mapsto |11\rangle$).