Assignment 4
Due date: November 13, 2014

For any part of a question, you will receive 25% of the grade (rounded up to an integer) if you write "I do not know how to answer this question" instead of an answer.

1. **General conversion from Stinespring form to Krauss form.** Suppose that you are given a description of a quantum operation that takes an n-qubit state ρ as input and produces an n'-qubit state σ as output, where the description is of the following form (where $n + m = n' + m'$):

   i. Append an m qubits, in state $|0^m\rangle$ to the end of the input state.
   ii. Apply an $(n + m)$-qubit unitary operation U.
   iii. Trace out the *first* m' qubits (resulting in an n'-qubit output).

   Show how to implement this in Krauss form as

$$\rho \mapsto \sum_{j \in S} A_j \rho A_j^\dagger,$$

   where $\sum_{j \in S} A_j^\dagger A_j = I$. Please be careful with the dimensions of your matrices/vectors (so that they make sense). Also, to avoid ambiguity between multiplication and tensor product, write $\otimes$ explicitly to denote the latter (it will be assumed that AB means the matrix product of A and B, as opposed to $A \otimes B$).

2. **Constructing an AND gate as a quantum operation.** Here we consider operations that map the two-qubit state $|a, b\rangle$ to the one-qubit state $|a \wedge b\rangle$, for all $a, b \in \{0, 1\}$. Of course, no unitary operation can perform this mapping, since the input and output dimension do not match; however, general quantum operations can compute this mapping.

   (a) Give k matrices $A_1, \ldots, A_k$ (where $k \leq 4$, and each A_j is a 2×4 matrix) such that $\sum_{j=1}^{k} A_j^\dagger A_j = I$ whose quantum operation computes the above mapping. In other words, for all $a, b \in \{0, 1\}$, when $\rho = |a, b\rangle\langle a, b|$,

$$\sum_{j=1}^{k} A_j \rho A_j^\dagger = |a \wedge b\rangle\langle a \wedge b|.$$

   (b) Your operation from part (a) maps all basis states to pure states. Does it map all pure input states to pure output states? Either prove the answer is yes, or provide a counterexample.

3. **Trace distance between pure states.**

   (a) Calculate an expression for the trace distance between $|0\rangle$ and $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ as a function of θ.

   (b) Calculate an expression for the Euclidean distance between the two points in the Bloch sphere that correspond to the pure states $|0\rangle$ and $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$.

4. **The density matrix is in the eye of the beholder.** Consider the following scenario. Alice first flips a biased coin that has outcome 0 with probability $\cos^2(\pi/8)$ and 1 with probability $\sin^2(\pi/8)$. If the coin value is 0 she creates the state $|0\rangle$ and if the coin value is 1 she creates the state $|1\rangle$. Then Alice sends the state that she created to Bob (she does not send the coin value).

   (a) From Alice's perspective (who *knows* the coin value), the density matrix of the state she created will be either $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. What is the density matrix of the state from Bob's perspective (who does *not* know the coin value)? Give the four matrix entries of this density matrix.

   (b) Suppose that, upon receiving the state from Alice, Bob measures it in the computational basis. The measurement process yields a classical bit and an output state ("collapsed" to $|0\rangle$ or $|1\rangle$). Will Bob's density matrix for the state (with Bob knowing the classical measurement outcome) be the same as Alice's?

   Suppose that we modify the above scenario to one where Alice flips a *fair* coin (where outcomes 0 and 1 each occur with probability 1/2) and if the coin value is 0 she creates the state $|\psi_0\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$ and if the coin value is 1 she creates the state $|\psi_1\rangle = \cos(\pi/8)|0\rangle - \sin(\pi/8)|1\rangle$. Alice sends the state (but not the coin value) to Bob.

   (c) From Alice's perspective (who *knows* the coin value), the density matrix of the state she created will be either $|\psi_0\rangle\langle\psi_0|$ or $|\psi_1\rangle\langle\psi_1|$. What is the density matrix of the state from Bob's perspective (who does *not* know the coin value)? Give the four matrix entries of this density matrix.

   (d) Suppose that, upon receiving the state from Alice, Bob measures it in the computational basis, yielding a classical bit and an output state ("collapsed" to $|0\rangle$ or $|1\rangle$). Bob knows the classical bit outcome from his measurement, but does not reveal this to Alice. Will Bob's density matrix for the output state be the same as Alice's?

5. **A nonlocal game.** Consider the game where Alice and Bob are physically separated and their goal is to produce outputs that satisfy the winning conditions specified below. Alice and Bob receive $s, t \in \{0, 1, 2\}$ as input (s to Alice and t to Bob), at which point they are forbidden from communicating with each other (so Alice has no idea what t is and Bob has no idea what s is). They each output a bit, a for Alice and b for Bob. The winning conditions are:

   - $a = b$ in the cases where $s = t$.
   - $a \neq b$ in the cases where $s \neq t$.

   (a) Show that, for any classical strategy of Alice and Bob, if it always succeeds in the $s = t$ cases, then it can succeed with probability at most $2/3$ in the $s \neq t$ cases.

   (b) Give a quantum strategy (that is, one where Alice and Bob can create an entangled state before the game starts and then base their outcomes on their measurements of their parts of this state) that always succeeds in the $s = t$ cases and succeeds with probability $3/4$ in the $s \neq t$ cases. (Hint: try the entangled state $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$ and have Alice and Bob perform rotations depending on s and t respectively.)