QIC710/CS768/CO681/PH767/AM871 Introduction to Quantum Information Processing (F14)

Assignment 1
Due date: September 25, 2014

1. **Distinguishing between pairs of quantum states.** In each case, one of the two given states is randomly selected (probability 1/2 each) and given to you. You are not told which one it is. Your goal is to guess which state was selected with as high a probability as you can achieve. Describe your distinguishing procedure as a unitary operation followed by a measurement (in the computational basis) and give its success probability. (Your assigned grade will depend on how close your distinguishing procedure is to optimal.)

   (a) $|0\rangle$ and $\cos\theta|0\rangle + \sin\theta|1\rangle$
   (where $\theta \in [0, \pi/2]$ is known to you, and your answer should be a function of θ)

   (b) $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $\frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)$ (where $i = \sqrt{-1}$)

   (c) **Optional for bonus credit:** In this case, there are three states,
   $|0\rangle$ and $-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ and $-\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$
   each occurring with probability 1/3, and the goal is to guess which one has been received. *Partial* bonus credit will be given for a distinguishing procedure that succeeds with probability at least $\frac{2}{3}\cos^2(\pi/12) \approx 0.662$. *Full* bonus credit will be given for a distinguishing procedure that succeeds with probability 2/3. (Warning: achieving 2/3 is tricky.)

2. **Entangled states and product states.** For each two-qubit state below, either express it as a product of two one-qubit states or show that such a factorization is impossible (in the latter case, the qubits are *entangled*).

   (a) $\frac{1}{2}|00\rangle + \frac{1}{2}i|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}i|11\rangle$

   (b) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

   (c) $\frac{9}{25}|00\rangle + \frac{12}{25}|01\rangle + \frac{12}{25}|10\rangle + \frac{16}{25}|11\rangle$

3. **Operations on part of an entangled quantum state.** Let $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. Prove *one* of the following three statements. (Hint: choose which one to prove carefully!)

   (a) For any 2×2 unitary matrix U, applying U to the first qubit of $|\psi\rangle$ has the same effect as applying U to the second qubit of $|\psi\rangle$.

   (b) For any 2×2 unitary matrix U, applying U to the first qubit of $|\psi\rangle$ has the same effect as applying U^T to the second qubit of $|\psi\rangle$ (U^T is the transpose of U).

   (c) For any 2×2 unitary matrix U, applying U to the first qubit of $|\psi\rangle$ has the same effect as applying $U^\dagger$ to the second qubit of $|\psi\rangle$ ($U^\dagger$ is the conjugate transpose of U).

4. **Constructing simple quantum circuits.**

(a) Describe a two-qubit quantum circuit consisting of *one* CNOT gate and *two* Hadamard gates that computes the following unitary tranformation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(b) Define the one-qubit gates H and S as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (\text{where } i = \sqrt{-1}).$$

In each case, give the 4×4 matrix corresponding to the two-qubit controlled gate:

•	H	•	S
H	•	S	•

(c) Give the 4×4 matrix corresponding to the following quantum circuit

H	•		×
	S	H	×

where S is as defined in part (b), and the last (two-qubit) gate denotes a *swap gate*, that transposes the two qubits (more precisely, a swap gate maps $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |10\rangle$, $|10\rangle \mapsto |01\rangle$, and $|11\rangle \mapsto |11\rangle$).

5. **Two-qubit quantum circuit with measurement on first qubit.** Consider the circuit

$\lvert 0\rangle$	R	•	R	$\mathcal{M}$
$\lvert\psi\rangle$		iZ		

where $|\psi\rangle$ is an arbitrary one-qubit state, the gate labeled $\mathcal{M}$ is a measurement,

$$R = \frac{1}{\sqrt{\cos\theta + \sin\theta}} \begin{pmatrix} \sqrt{\cos\theta} & \sqrt{\sin\theta} \\ \sqrt{\sin\theta} & -\sqrt{\cos\theta} \end{pmatrix} \quad \text{and} \quad iZ = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

(a) What is the probability that the outcome of the measurement of the first qubit is 0?

(b) Conditional on the outcome of the measurement of the first qubit being 0, what is the transformation applied to the second qubit? (It is expressible as a 2×2 matrix.)