QIC710/CS678/CO681/PH767/AM871 Introduction to Quantum Information Processing (F13)

Assignment 4
Due date: November 7, 2013

1. **Trace distance between pure states.**
   (a) Calculate an expression for the trace distance between $|0\rangle$ and $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ as a function of θ.
   (b) Calculate an expression for the Euclidean distance between the two points in the Bloch sphere that correspond to the pure states $|0\rangle$ and $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$.
   (c) Repeat parts (a) and (b) for these two states: $\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and the pure state $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$. Give the trace distance between the two states and also the Euclidean distance between the two points on the Bloch sphere.

2. **The density matrix is in the eye of the beholder.** Consider the following scenario. Alice first flips a biased coin that has outcome 0 with probability $\cos^2(\pi/8)$ and 1 with probability $\sin^2(\pi/8)$. If the coin value is 0 she creates the state $|0\rangle$ and if the coin value is 1 she creates the state $|1\rangle$. Then Alice sends the state that she created to Bob (she does not send the coin value).

   (a) From Alice's perspective (who *knows* the coin value), the density matrix of the state she created will be either $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. What is the density matrix of the state from Bob's perspective (who does *not* know the coin value)? Give the four matrix entries of this density matrix.
   (b) Suppose that, upon receiving the state from Alice, Bob measures it in the computational basis. The measurement process yields a classical bit and an output state ("collapsed" to $|0\rangle$ or $|1\rangle$). Will Bob's density matrix for the state (with Bob knowing the classical measurement outcome) be the same as Alice's?

   Suppose that we modify the above scenario to one where Alice flips a *fair* coin (where outcomes 0 and 1 each occur with probability 1/2) and if the coin value is 0 she creates the state $|\psi_0\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$ and if the coin value is 1 she creates the state $|\psi_1\rangle = \cos(\pi/8)|0\rangle - \sin(\pi/8)|1\rangle$. Alice sends the state (but not the coin value) to Bob.

   (c) From Alice's perspective (who *knows* the coin value), the density matrix of the state she created will be either $|\psi_0\rangle\langle\psi_0|$ or $|\psi_1\rangle\langle\psi_1|$. What is the density matrix of the state from Bob's perspective (who does *not* know the coin value)? Give the four matrix entries of this density matrix.
   (d) Suppose that, upon receiving the state from Alice, Bob measures it in the computational basis, yielding a classical bit and an output state ("collapsed" to $|0\rangle$ or $|1\rangle$). Bob knows the classical bit outcome from his measurement, but does not reveal this to Alice. Will Bob's density matrix for the output state be the same as Alice's?

3. **General conversion from Stinespring form to Krauss form.** Suppose that you are given a description of a quantum operation that takes an n-qubit state ρ as input and produces an n'-qubit state σ as output, where the description is of the following form (where $n + m = n' + m'$):

   i. Append an m qubits, in state $|0^m\rangle$ to the end of the input state.
   ii. Apply an $(n + m)$-qubit unitary operation U.
   iii. Trace out the *first* m' qubits (resulting in an n'-qubit output).

   Show how to implement this in Krauss form as

$$\rho \mapsto \sum_{j \in S} A_j \rho A_j^\dagger,$$

   where $\sum_{j \in S} A_j^\dagger A_j = I$. Please be careful with the dimensions of your matrices/vectors (so that they make sense). Also, to avoid ambiguity between multiplication and tensor product, write $\otimes$ explicitly to denote the latter (it will be assumed that AB means the matrix product of A and B, as opposed to $A \otimes B$).

4. **Analysis of a particular quantum operation.** Let p be an arbitrary real-valued parameter such that $0 < p < 1$. We will explore some nice properties of the one-qubit operation defined by the two Krauss operators

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}. \tag{1}$$

   It is easy to verity that $A_0^\dagger A_0 + A_1^\dagger A_1 = I$, so the operation D_p that maps each 2×2 density matrix ρ to

$$D_p(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger \tag{2}$$

   is indeed a valid quantum operation.

   (a) Find a state that is a *fixed point* of D_p. A fixed point is a one-qubit density matrix ρ_0 such that $D_p(\rho_0) = \rho_0$.

   (b) Show that the operation $D_p^{(2)}$, corresponding to applying D_p twice in succession (that is, $D_p^{(2)}(\rho) = D_p(D_p(\rho))$ is equivalent to applying D_q once for some suitably chosen value of q. Give an expression for q as a function of p.

   (c) Generalizing part (b), we can also define the operation $D_p^{(k)}$, corresponding to applying D_p k times in succession. What are the Krauss operators of $D_p^{(k)}$? Specify their matrix entries with closed-form expressions in terms of p and k.

   (d) Is $\lim_{k\to\infty} D^{(k)}(\rho) = \rho_0$ for *any* initial state ρ, where ρ_0 is the fixed point of D_p that you gave in part (a)?

Question 5 is on the next page

5. **Constructing an AND gate as a quantum operation.** Here we consider operations that map the two-qubit state $|a, b\rangle$ to the one-qubit state $|a \wedge b\rangle$, for all $a, b \in \{0, 1\}$. Of course, no unitary operation can perform this mapping, since the input and output dimension do not match; however, general quantum operations can compute this mapping.

   (a) Give k matrices $A_1, \ldots, A_k$ (where $k \leq 4$, and each A_j is a 2×4 matrix) such that $\sum_{j=1}^{k} A_j^\dagger A_j = I$ whose quantum operation computes the above mapping. In other words, for all $a, b \in \{0, 1\}$, when $\rho = |a, b\rangle\langle a, b|$,

   $$\sum_{j=1}^{k} A_j \rho A_j^\dagger = |a \wedge b\rangle\langle a \wedge b|.$$

   (b) Your operation from part (a) maps all basis states to pure states. Does it map all pure input states to pure output states? Either prove the answer is yes, or provide a counterexample.