QIC710/CS667/CO681/PH767/AM871 Introduction to Quantum Information Processing (F12)

Assignment 1
Due date: September 25, 2012

1. **Distinguishing between pairs of quantum states.** In each case, one of the two given states is selected according to the uniform distribution and given to you. You do not know which one. Your goal is to guess which state was selected with as high a probability as you can achieve. Describe your distinguishng procedure as a unitary operation followed by a measurement (in the computational basis) and give its success probability. (Your assigned grade will depend on how close your distinguishing procedure is to optimal.)

   (a) $|0\rangle$ and $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   (b) $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ (where $i = \sqrt{-1}$)

   (c) $\cos\theta|0\rangle - \sin\theta|1\rangle$ and $\cos\theta|0\rangle + \sin\theta|1\rangle$ (where $\theta \in [0, 2\pi)$ is known to you)

   (d) **Optional for bonus credit:** (This may be a challenge for people new to quantum information.) In this variant of the problem, $|\psi\rangle$ is an arbitrary one-qubit state that you know absolutely nothing about. You are either given a "same" pair of qubits or an "orthogonal" pair of qubits, and each scenario occurs with probability 1/2. In the case of a "same" pair, you are given the state $|\psi\rangle|\psi\rangle$ (two copies of $|\psi\rangle$). In the case of an "orthogonal" pair, you are given the state $|\psi\rangle|\phi\rangle$, where $|\phi\rangle$, is any state orthogonal to $|\psi\rangle$ (i.e., $\langle\psi|\phi\rangle = 0$). (You know nothing about $|\phi\rangle$ except that it is orthogonal to $|\psi\rangle$.) Your goal is to guess whether you have a "same" pair or an "orthogonal" pair with as high a success probability as possible. Whatever your strategy achieves should be for whatever the unknown $|\psi\rangle$ and $|\phi\rangle$ happen to be.

2. **Entangled states and product states.** For each two-qubit state below, either express it as a product of two one-qubit states or show that such a factorization is impossible (in the latter case, the qubits are *entangled*).

   (a) $\frac{1}{2}|00\rangle + \frac{1}{2}i|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}i|11\rangle$

   (b) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

   (c) $\frac{9}{25}|00\rangle + \frac{12}{25}|01\rangle + \frac{12}{25}|10\rangle + \frac{16}{25}|11\rangle$

3. **Operations on part of an entangled quantum state.** Let $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. Prove *one* of the following three statements.

   (a) For any 2×2 unitary matrix U, applying U to the first qubit of $|\psi\rangle$ has the same effect as applying U to the second qubit of $|\psi\rangle$.

   (b) For any 2×2 unitary matrix U, applying U to the first qubit of $|\psi\rangle$ has the same effect as applying U^T to the second qubit of $|\psi\rangle$ (U^T is the transpose of U).

   (c) For any 2×2 unitary matrix U, applying U to the first qubit of $|\psi\rangle$ has the same effect as applying $U^\dagger$ to the second qubit of $|\psi\rangle$ ($U^\dagger$ is the conjugate transpose of U).

4. **Constructing simple quantum circuits.**

   (a) Describe a two-qubit quantum circuit consisting of *one* CNOT gate and *two* Hadamard gates that computes the following unitary tranformation:

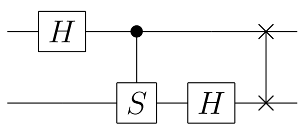
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

   (b) Define the one-qubit gates H and S as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (\text{where } i = \sqrt{-1}).$$

   In each case, give the 4×4 matrix corresponding to the two-qubit controlled gate:

   (c) Give the 4×4 matrix corresponding to the following quantum circuit



   where S is as defined in part (b), and the last (two-qubit) gate denotes a *swap gate*, that transposes the two qubits (more precisely, a swap gate maps $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |10\rangle$, $|10\rangle \mapsto |01\rangle$, and $|11\rangle \mapsto |11\rangle$).

5. **Distinguishing between a set of "tetrahedral" states.** Consider the following four states (note that they are confined to a 3-dimensional space):

$$\begin{aligned}
|\psi_0\rangle &= \tfrac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle) \\
|\psi_1\rangle &= \tfrac{1}{\sqrt{3}}(|00\rangle - |01\rangle - |10\rangle) \\
|\psi_2\rangle &= \tfrac{1}{\sqrt{3}}(-|00\rangle + |01\rangle - |10\rangle) \\
|\psi_3\rangle &= \tfrac{1}{\sqrt{3}}(-|00\rangle - |01\rangle + |10\rangle).
\end{aligned}$$

   It is easy to check that the inner product between each pair is $-1/3$, and they can be viewed geometrically as corresponding to the vertices of a regular tetrahedron in three dimensions. Suppose one of these states is uniformly selected and sent to you, and your goal is to guess which state it is. Describe a procedure based on unitary operations and measurements on the two-qubit system that predicts the state with as high a success probability as you can achieve. (Your assigned grade will depend on how close your procedure is to optimal.)