QIC710/CS667/CO681/PH767/AM871 Introduction to Quantum Information Processing (F11)

Assignment 4
Due date: November 15, 2011

1. **On distinguishing between an identical pair and an orthogonal pair of qubits.** Let α, β be any complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$. Consider the state distinguishing problem where we are given two qubits whose joint state is either

$$(\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \qquad \text{or} \qquad (\alpha|0\rangle + \beta|1\rangle) \otimes (\bar{\beta}|0\rangle - \bar{\alpha}|1\rangle)$$

and our goal is to determine which case we are in. Call the cases "same" and "orthogonal" (noting that $\alpha|0\rangle + \beta|1\rangle$ and $\bar{\beta}|0\rangle - \bar{\alpha}|1\rangle$ are orthogonal). The key complication here is that: *we are given absolutely no information about α and β, not even a probability distribution from which they arise.* Our approach must succeed with probabiity $\frac{1}{2} + \varepsilon$ *whatever* α and β are and *whatever* case we are in.

We will use the Bell basis (which is an orthonormal basis): $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

   (a) Show that, in the case "same", the state is in the subspace spanned by the first three Bell states: $|\phi^+\rangle$, $|\phi^-\rangle$, and $|\psi^+\rangle$.

   (b) Show that, in the case "orthogonal", there exists a constant δ (where δ is independent of what α and β are) such that the absolute value of the inner product of the state with $|\psi^-\rangle$ is δ. Also, indicate what δ is.

   (c) Based on your answers in (a) and (b), give a distinguishing procedure that, whatever case and whatever α and β are, will correctly identify the case with probability $\frac{1}{2} + \varepsilon$, for some positive constant ε as large as possible.

2. **Operations, states and the Bloch sphere.** Consider the unitary matrices of the form

$$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

where $\theta \in [0, 2\pi)$.

   (a) Prove that, for any value of θ, M_θ is a *reflection*. In other words, that the eigenvalues of M_θ are from $\{+1, -1\}$. (Note: there is a very easy way of doing this.)

   (b) Prove that, on the Bloch sphere, M_θ acts as a *rotation* (for any value of θ). Give the axis of rotation and angle of rotation.

   (c) Specify four *one-qubit* quantum states $|\phi_0\rangle$, $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$, such that, for all $j \neq k$, $|\langle\phi_j|\phi_k\rangle| \leq r$, for as small an r as possible. Note that, using states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$, achieves $r = 1/\sqrt{2}$, but a smaller r is achievable. You may specify the states by their density matrices; however, you should explicitly give the value of r obtained.

3. **Unitary that always maps every state to an orthogonal state?** Is there a one-qubit unitary operation that maps each pure state $|\psi\rangle$ to some state $|\psi'\rangle$ such that $\langle\psi|\psi'\rangle = 0$? If so, specify the unitary operation. If not, prove that no such unitary operation exists.

4. **Constructing an AND gate as a quantum operation.** Here we consider operations that map the two-qubit state $|a, b\rangle$ to the one-qubit state $|a \wedge b\rangle$, for all $a, b \in \{0, 1\}$. Of course, no unitary operation can perform this mapping, since the input and output dimension do not match; however, general quantum operations can compute this mapping.

   (a) Give four 2×4 matrices A_1, A_2, A_3, A_4 such that $\sum_{j=1}^{4} A_j^\dagger A_j = I$ that compute the above mapping in that, for all $a, b \in \{0, 1\}$, when $\rho = |a, b\rangle\langle a, b|$,

$$\sum_{j=1}^{4} A_j \rho A_j^\dagger = |a \wedge b\rangle\langle a \wedge b|.$$

   (b) Your operation from part (a) maps all basis states to pure states. Does it map all pure input states to pure output states? Either prove the answer is yes, or provide a counterexample.

   (c) Here we implement the mapping in the Stinespring form, using additional qubits at the beginning, performing a unitary operation, and then tracing out qubits. Consider this three-step process (where the input is any two-qubit quantum state):

      i. Append a third qubit in state $|0\rangle$ to the end of the two input qubits.
      ii. Apply a 3-qubit unitary operation U.
      iii. Trace out the second and third qubit (resulting in a single qubit, taken as the output).

   Describe a 3-qubit unitary U that causes this process to implement the mapping above (that is, to map $|a, b\rangle$ to $|a \wedge b\rangle$, for all $a, b \in \{0, 1\}$).

5. **General conversion from Stinespring form to Krauss form.** Suppose that you are given a description of a quantum operation that takes an n-qubit state ρ as input and produces an n'-qubit state σ as output, where the description is of the following form (where $n + m = n' + m'$):

   i. Append an m qubits, in state $|0^m\rangle$ to the end of the input state.
   ii. Apply an $(n + m)$-qubit unitary operation U.
   iii. Trace out the *first* m' qubits (resulting in an n'-qubit output).

   Show how to implement this in Krauss form as

$$\rho \mapsto \sum_{j \in S} A_j \rho A_j^\dagger,$$

   where $\sum_{j \in S} A_j^\dagger A_j = I$. Please be careful with the dimensions of your matrices/vectors (so that they make sense). Also, to avoid ambiguity between multiplication and tensor product, write $\otimes$ explicitly to denote the latter (I will assume that AB means the matrix product of A and B, as opposed to $A \otimes B$).