QIC710/CS667/CO681/PH767/AM871 Introduction to Quantum Information Processing (F11)

Assignment 1
Due date: September 27, 2011

1. **Distinguishing between pairs of quantum states.** In each case, one of the two given states is selected according to the uniform distribution and given to you. You do not know which one. Your goal is to guess which state was selected with as high a probability as you can achieve. Describe your distinguishng procedure as a unitary operation followed by a measurement (in the computational basis) and give its success probability. (Your assigned grade will depend on how close your distinguishing procedure is to optimal.)

   (a) $|0\rangle$ and $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   (b) $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ (where $i = \sqrt{-1}$)

   (c) $\cos\theta|0\rangle - \sin\theta|1\rangle$ and $\cos\theta|0\rangle + \sin\theta|1\rangle$ (where $\theta \in [0, 2\pi)$ is known to you)

   (d) **Optional for bonus credit:** In this case, there are three states,

   $|0\rangle$ and $-\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ and $-\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$

   each occurring with probability 1/3, and the goal is to guess which one has been received. *Partial* bonus credit will be given for a distinguishing procedure that succeeds with probability at least $\frac{2}{3}\cos^2(\pi/12) \approx 0.662$. *Full* bonus credit will be given for a distinguishing procedure that succeeds with probability 2/3, which is the best possible. (Warning: achieving 2/3 is tricky.)

2. **Entangled states and product states.** For each two-qubit state below, either express it as a product of two one-qubit states or show that such a factorization is impossible (in the latter case, the qubits are *entangled*).

   (a) $\frac{1}{2}|00\rangle + \frac{1}{2}i|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}i|11\rangle$

   (b) $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

   (c) $\frac{9}{25}|00\rangle + \frac{12}{25}|01\rangle + \frac{12}{25}|10\rangle + \frac{16}{25}|11\rangle$

3. **Operations on part of an entangled quantum state.** Let

   $$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

   Prove *one* of the following three statements.

   (a) For any 2×2 unitary matrix U, applying U to the first qubit of $|\psi\rangle$ has the same effect as applying U to the second qubit of $|\psi\rangle$.

   (b) For any 2×2 unitary matrix U, applying U to the first qubit of $|\psi\rangle$ has the same effect as applying U^T to the second qubit of $|\psi\rangle$ (U^T is the transpose of U).

   (c) For any 2×2 unitary matrix U, applying U to the first qubit of $|\psi\rangle$ has the same effect as applying $U^\dagger$ to the second qubit of $|\psi\rangle$ ($U^\dagger$ is the conjugate transpose of U).

4. **Constructing simple quantum circuits.**

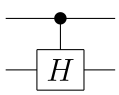
(a) Describe a two-qubit quantum circuit consisting of *one* CNOT gate and *two* Hadamard gates that computes the following unitary tranformation:

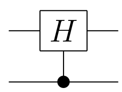
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

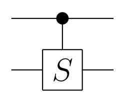
(b) Define the one-qubit gates H and S as

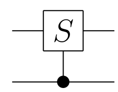
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (\text{where } i = \sqrt{-1}).$$

In each case, give the 4×4 matrix corresponding to the two-qubit controlled gate:

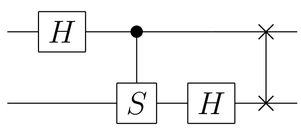








(c) Give the 4×4 matrix corresponding to the following quantum circuit



where S is as defined in part (b), and the last (two-qubit) gate denotes a *swap gate*, that transposes the two qubits (more precisely, a swap gate maps $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |10\rangle$, $|10\rangle \mapsto |01\rangle$, and $|11\rangle \mapsto |11\rangle$).

5. **Distinguishing between the tetrahedral qutrit states.** Consider the following four qutrit (quantum trit) states:

$$\begin{aligned} |\psi_0\rangle &= \tfrac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle) \\ |\psi_1\rangle &= \tfrac{1}{\sqrt{3}}(|0\rangle - |1\rangle - |2\rangle) \\ |\psi_2\rangle &= \tfrac{1}{\sqrt{3}}(-|0\rangle + |1\rangle - |2\rangle) \\ |\psi_3\rangle &= \tfrac{1}{\sqrt{3}}(-|0\rangle - |1\rangle + |2\rangle). \end{aligned}$$

It is easy to check that the inner product between each pair is $-1/3$, and they can be viewed geometrically as corresponding to the vertices of a regular tetrahedron. Suppose one of these states is uniformly selected and sent to you. Describe a procedure based on unitary operations and measurements (possibly in a larger Hilbert space) that predicts the state with as high a success probability as you can achieve. (Your assigned grade will depend on how close your procedure is to optimal.)