CS667/CO681/PH767 Quantum Information Processing (Fall 08)

Assignment 4

Due date: November 13, 2008

1. **Basic properties of density matrices.**

   (a) A density matrix ρ corresponds to a *pure* state if and only if $\rho = |\psi\rangle\langle\psi|$. Show that ρ corresponds to a pure state if and only if $\mathrm{Tr}(\rho^2) = 1$.

   (b) Show that for any operator ρ that is Hermitian, positive semidefinite (i.e., no negative eigenvalues), and has trace 1, there is a probabilistic mixture of pure states whose density matrix is ρ.

2. **A tripartite entangled state.** Consider the three-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$.

   (a) Give the density matrix of the first two qubits of the above state. Are the first two qubits entangled?

   (b) Suppose that the three qubits of $|\psi\rangle$ are in three physically separated labs, owned by Alice, Bob, and Carol. Suppose that Carol measures her qubit in the basis $\{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\}$ and then broadcasts a bit corresponding to the classical outcome of the measurement to Alice and Bob. Show that, based on this bit, Alice and Bob can each independently apply a unitary transformation to the qubit in their own lab so that the resulting two-qubit state is entangled.

3. **Constructing a specific general quantum operation.**

   (a) Give a general quantum operation in the Krauss form that maps each two-qubit basis state to another basis state as follows. For all $a, b \in \{0, 1\}$, $|a\rangle|b\rangle \mapsto |a\rangle|a \wedge b\rangle$. Note that this mapping cannot be implemented by a unitary operation, since it is not reversible. The goal is to give a mapping of the form $\rho \mapsto \sum_{k=1}^{m} A_k \rho A_k^\dagger$, where $\sum_{k=1}^{m} A_k^\dagger, A_k = I$ that acts on the basis states as above.

   (b) What quantum state does the operation in part (a) map $\frac{1}{\sqrt{2}}|1\rangle(|0\rangle + |1\rangle)$ to? (You can describe the state in terms of its density matrix.)

4. **Is the transpose a valid quantum operation?** Here we consider an operation on qubits that we denote by Λ, defined as $\Lambda(\rho) = \rho^T$ for each density matrix ρ (where ρ^T is the transpose of T).

   (a) Give an example of a one-qubit pure state $|\psi\rangle$ such that $\Lambda(|\psi\rangle\langle\psi|)$ is a pure state orthogonal to $|\psi\rangle$.

   (b) Show that Λ is not a valid quantum operation in that there is no general quantum operation, of the form $\rho \mapsto \sum_{k=1}^{m} A_k \rho A_k^\dagger$, where $\sum_{k=1}^{m} A_k^\dagger A_k = I$ that implements Λ. For this question, you may use without proof the fact that any general quantum operation can be extended so that it acts on part of a larger system. More precisely, you may assume that if Λ is a valid one-qubit quantum operation then there is a valid two-qubit quantum operation that, for all pairs of density matrices σ and ρ, maps $\sigma \otimes \rho$ to $\sigma \otimes \Lambda(\rho)$.

(Hint: consider how this extended operation would act on the state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.)

5. **Searching when the density of marked items is $1/4$ and $1/2$.**

   (a) Suppose that $f : \{0,1\}^n \to \{0,1\}$ has the property that for exactly $\frac{1}{4}2^n$ of the values of $x \in \{0,1\}^n$, $f(x) = 1$ and the goal is to find such an x. Show that Grover's algorithm is guaranteed to find such an x after a single iteration (and thus with a single query to U_f).

   (b) The same question as part (a), except assume that f has the property that for exactly $\frac{1}{2}2^n$ of the values of $x \in \{0,1\}^n$, $f(x) = 1$. Can such an x still be found with a single query to U_f?