Solutions to Sample Assignment A

1(a) What is the most general form of a strategy for this game? For each input received (0 or 1), there can be some probability distribution for the resulting guess. However, in the case of input 1 guessing I is *never* correct. Therefore, no generality is lost if we restrict our attention to strategies of this form (for some parameter $\epsilon \in [0,1]$):

Strategy B(ϵ)

if the bit received is 0 then randomly guess $\begin{cases} \text{I} & \text{with probability } 1-\epsilon \\ \text{II} & \text{with probability } \epsilon \end{cases}$

if the bit received is 1 then guess II

(Any strategy that is not of this form can be improved by changing it to always guess II in the case of input 1.)

For set-up I, this strategy succeeds with probability $1-\epsilon$. For set-up II, this strategy succeeds with probability $\frac{1}{2}+\frac{1}{2}\epsilon$. Therefore, the average-case success probability of strategy B(ϵ) is the average of these two success probabilities

$$\tfrac{1}{2}(1-\epsilon)+\tfrac{1}{2}(\tfrac{1}{2}+\tfrac{1}{2}\epsilon) = \tfrac{3}{4}-\tfrac{1}{2}\epsilon+\tfrac{1}{4}\epsilon \quad (1)$$

$$= \tfrac{3}{4}-\tfrac{1}{4}\epsilon. \quad (2)$$

This is maximized when $\epsilon = 0$. Therefore, strategy B(0), which is the same as strategy A, achieves optimum average-case success probability.

Note: It was mentioned in class that, in the context of average-case success probability, we need only consider the four deterministic strategies and note that strategy A is the best among these. This is a consequence of a general result that's sometimes referred to as "Yao's Lemma". A solution along these lines would need to include a clear statement or reference to that lemma. Note that there is no such lemma for worst-case success probability; in fact, in part (b) the optimal strategy is *not* deterministic.

1(b) The highest possible worst-case success probability is $\frac{2}{3}$ (higher than what strategy A attains). To see why this is so, we refer to the strategies of the form B(ϵ) that we defined in part (a) (without loss of generality, we need only consider strategies of the form B(ϵ)). The worst-case success probability of B(ϵ) is the minimum of the success probabilities of the two cases, which is

$$\min\left\{1-\epsilon,\ \tfrac{1}{2}+\tfrac{1}{2}\epsilon\right\}. \quad (3)$$

For what value of parameter ϵ is this maximized? Since $1-\epsilon$ decreases as a function of ϵ and $\frac{1}{2}+\frac{1}{2}\epsilon$ increases as a function of ϵ, the expression is maximized at the value of ϵ where the expressions are equal

$$1-\epsilon = \tfrac{1}{2}+\tfrac{1}{2}\epsilon, \quad (4)$$

which occurs when $\epsilon = \frac{1}{3}$. Therefore, worst-case success probability is maximized by strategy B($\frac{1}{3}$) and is $\frac{2}{3}$.

2. If the rotation R_θ is applied then the two states become

$$R_\theta|0\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle \quad \text{vs.} \quad R_\theta|+\rangle = \cos(\theta + \tfrac{\pi}{4})|0\rangle + \sin(\theta + \tfrac{\pi}{4})|1\rangle. \tag{5}$$

Since $\theta \in [0, \pi/4]$, $R_\theta|0\rangle$ is closer to $|0\rangle$ than $|1\rangle$, and $R_\theta|+\rangle$ is closer to $|1\rangle$ than $|0\rangle$. Therefore, we can assume that: for measurement outcome 0 the guess is 0; and for outcome 1 the guess is + (e.g., it's easy to check that guessing + in case of outcome 0 would lead to a lower success probability).

The average-case success probability as a function of θ is

$$p(\theta) = \tfrac{1}{2}|\langle 0|R_\theta|0\rangle|^2 + \tfrac{1}{2}|\langle 1|R_\theta|+\rangle|^2 = \tfrac{1}{2}\cos^2(\theta) + \tfrac{1}{2}\sin^2(\theta + \tfrac{\pi}{4}). \tag{6}$$

For what $\theta \in [0, \pi/4]$ is this maximized? Since $p(\theta)$ is differentable, we can use calculus to determine where the maximum is. The derivative is

$$\begin{aligned} p'(\theta) &= \cos(\theta)\sin(\theta) - \sin(\theta + \tfrac{\pi}{4})\cos(\theta + \tfrac{\pi}{4}) && (7)\\ &= \tfrac{1}{2}\sin(2\theta) - \tfrac{1}{2}\sin(2(\theta + \tfrac{\pi}{4})), && (8) \end{aligned}$$

where we have used the formula $\sin(2x) = 2\sin(x)\cos(x)$ in Eq. (8).

The derivative is zero when

$$\sin(2\theta) = \sin(2(\theta + \tfrac{\pi}{4})). \tag{9}$$

Since $\theta = \theta + \frac{\pi}{4}$ cannot occur, the only way for Eq. (9) to be satisfied is if

$$2(\theta + \tfrac{\pi}{4}) = \pi - 2\theta \tag{10}$$

(using the fact that $\sin(x) = \sin(\pi - x)$). This occurs if and only if $\theta = \pi/8$. Therefore the optimal rotation angle $\theta \in [0, \frac{\pi}{4}]$ must be one of $0, \frac{\pi}{8}, \frac{\pi}{4}$. It's easy to check these three cases and $\theta = \frac{\pi}{8}$ is the optimum. For $\theta = \frac{\pi}{8}$, the average-case success probability is

$$\begin{aligned} p(\tfrac{\pi}{8}) &= \tfrac{1}{2}\cos^2(\tfrac{\pi}{8}) + \tfrac{1}{2}\sin^2(\tfrac{\pi}{8} + \tfrac{\pi}{4}) && (11)\\ &= \cos^2(\tfrac{\pi}{8}). && (12) \end{aligned}$$

Note: For this state distinguishing problem, $\cos^2(\frac{\pi}{8})$ is also the highest worst-case success probability.

3. Suppose that

$$\begin{aligned} \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle &= (\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle) \\ &= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle. && (13) \end{aligned}$$

We'll show that this results in a contradiction. Eq. (13) implies

$$\begin{aligned} \alpha_0\beta_0 &= \tfrac{1}{\sqrt{2}} && (14)\\ \alpha_0\beta_1 &= 0 && (15)\\ \alpha_1\beta_0 &= 0 && (16)\\ \alpha_1\beta_1 &= \tfrac{1}{\sqrt{2}}. && (17) \end{aligned}$$

There is no solution to these equations because Eq. (15) implies that either $\alpha_0 = 0$ (which contradicts Eq. (14)) or $\beta_1 = 0$ (which contradicts Eq. (17)).