Preliminary remarks about quantum communication
Quantum information can apparently be used to substantially reduce *computation* costs for a number of interesting problems.

How does quantum information affect the *communication costs* of information processing tasks?

We explore this issue ...
Entanglement and signaling

Recall that Entangled states, such as \( \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \),

can be used to perform some intriguing feats, such as teleportation and superdense coding

—but they cannot be used to “signal instantaneously”

Any operation performed on one system has no affect on the state of the other system (its reduced density matrix)
Basic communication scenario

Goal: convey $n$ bits from Alice to Bob

$x_1 x_2 \ldots x_n$

Alice

$x_1 x_2 \ldots x_n$

Bob

Resources
Basic communication scenario

Bit communication:

Cost: \( n \)

Qubit communication:

Cost: \( n \) [Holevo’s Theorem, 1973]

Bit communication & prior entanglement:

Cost: \( n \) (can be deduced)

Qubit communication & prior entanglement:

Cost: \( n/2 \) superdense coding [Bennett & Wiesner, 1992]
The GHZ “paradox”
(Greenberger-Horne-Zeilinger)
Rules of the game:
1. It is promised that \( r \oplus s \oplus t = 0 \)
2. No communication after inputs received
3. They win if \( a \oplus b \oplus c = r v s v t \)
No perfect strategy for GHZ

Input: \( r \)

Output: \( a \)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
rst & a \oplus b \oplus c \\
\hline
000 & 0 \\
011 & 1 \\
101 & 1 \\
110 & 1 \\
\hline
\end{array}
\]

General deterministic strategy:
\[
a_0, a_1, b_0, b_1, c_0, c_1
\]

Winning conditions:
\[
\begin{cases}
a_0 \oplus b_0 \oplus c_0 = 0 \\
a_0 \oplus b_1 \oplus c_1 = 1 \\
a_1 \oplus b_0 \oplus c_1 = 1 \\
a_1 \oplus b_1 \oplus c_0 = 1 \\
\end{cases}
\]

Has no solution, thus no perfect strategy exists
Input and output events can be \textit{space-like} separated: so signals at the speed of light are not fast enough for cheating.

What if Alice, Bob, and Carol \textit{still} keep on winning?
To be continued …
Continuation of:
The GHZ “paradox”
(Greenberger-Horne-Zeilinger)
"GHZ Paradox" explained

Prior entanglement: $|\psi\rangle = |000\rangle - |011\rangle - |101\rangle - |110\rangle$

Alice’s strategy:
1. if $r = 1$ then apply $H$ to qubit (else $I$)
2. measure qubit and set $a$ to result

Bob’s & Carol’s strategies: similar

Case 1 ($rst = 000$): state is measured directly … 😥

Case 2 ($rst = 011$): new state $|001\rangle + |010\rangle - |100\rangle + |111\rangle$ 😖

Cases 3 & 4 ($rst = 101$ & 110): similar by symmetry 😊
GHZ: conclusions

- For the GHZ game, any classical team succeeds with probability at most $\frac{3}{4}$

- Allowing the players to communicate would enable them to succeed with probability 1

- Entanglement cannot be used to communicate

- Nevertheless, allowing the players to have entanglement enables them to succeed with probability 1 (but not by using entanglement to communicate)

- Thus, entanglement is a useful resource for the task of winning the GHZ game
The Bell inequality and its violation – Physicist’s perspective
Bell’s Inequality and its violation

Part I: physicist’s view:

Can a quantum state have *pre-determined* outcomes for each possible measurement that can be applied to it?

qubit: where the “manuscript” is something like this:

called *hidden variables*

[Bell, 1964]
[Clauser, Horne, Shimony, Holt, 1969]
Imagine a two-qubit system, where one of two measurements, called $M_0$ and $M_1$, will be applied to each qubit:

Define:

$A_0 = (-1)^a_0$
$A_1 = (-1)^a_1$
$B_0 = (-1)^b_0$
$B_1 = (-1)^b_1$

Claim: $A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 \leq 2$

Proof: $A_0(B_0 + B_1) + A_1(B_0 - B_1) \leq 2$

one is $\pm 2$ and the other is 0
Bell Inequality

\[ A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2 \] is called a **Bell Inequality**

**Question:** could one, in principle, design an experiment to check if this Bell Inequality holds for a particular system?

**Answer 1:** *no, not directly*, because \( A_0, A_1, B_0, B_1 \) cannot all be measured (only one \( A_s B_t \) term can be measured)

**Answer 2:** *yes, indirectly*, by making many runs of this experiment: pick a random \( st \in \{00, 01, 10, 11\} \) and then measure with \( M_s \) and \( M_t \) to get the value of \( A_s B_t \)

The **average** of \( A_0 B_0, A_0 B_1, A_1 B_0, -A_1 B_1 \) should be \( \leq \frac{1}{2} \)

* also called CHSH Inequality
Recap of Bell Inequality

Consider the following experiment:

1. pick a random \( st \in \{00, 01, 10, 11\} \) (uniform distribution)
2. perform \( M_s \) measurement on 1\(^{st}\) qubit (outcome \( A_s \in \{+1, -1\} \))
3. perform \( M_t \) measurement on 2\(^{nd}\) qubit (outcome \( B_t \in \{+1, -1\} \))
4. output the value of \((-1)^{s \cdot t} A_s B_t\)

In any run of this experiment, the output is an element of \( \{+1, -1\} \)
(according to probabilities that depend on what \( A_0, A_1, B_0, B_1 \) are)

How large can the expected value of the outcome be?

\[
\frac{1}{4} (A_0 B_0) + \frac{1}{4} (A_0 B_1) + \frac{1}{4} (A_1 B_0) + \frac{1}{4} (-A_1 B_1) \\
= \frac{1}{4} (A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1) \leq \frac{1}{4} 2 = \frac{1}{2}
\]
Violating the Bell Inequality

Assume the quantum mechanical framework is correct

Two-qubit system in state

\[ |\phi\rangle = |00\rangle - |11\rangle \]

It can be shown that, applying rotations \(\theta_A\) and \(\theta_B (R_{\theta_A} \otimes R_{\theta_B})\) yields:

\[ \cos(\theta_A + \theta_B) (|00\rangle - |11\rangle) + \sin(\theta_A + \theta_B) (|01\rangle + |10\rangle) \]

Define

- \(M_0\): rotate by \(-\pi/16\) then measure
- \(M_1\): rotate by \(+3\pi/16\) then measure

Then \(A_0B_0, A_0B_1, A_1B_0, -A_1B_1\) all have expected value \(1/2\sqrt{2}\), which contradicts the upper bound of \(1/2\)

Therefore, QM framework implies LHV framework is false
Bell Inequality violation: summary

Assuming that quantum systems are governed by \textit{local hidden variables} leads to the Bell inequality
\[
A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2
\]

But this is \textit{violated} in the case of Bell states (by a factor of $\sqrt{2}$)

Therefore, no such hidden variables exist

This is, in principle, experimentally verifiable, and experiments along these lines have actually been conducted
The Bell inequality and its violation – Computer Scientist’s perspective
Bell’s Inequality and its violation

Part II: computer scientist’s view:

input: 

<table>
<thead>
<tr>
<th>input</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
</tr>
<tr>
<td>$t$</td>
</tr>
</tbody>
</table>

output: 

<table>
<thead>
<tr>
<th>output</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
</tr>
<tr>
<td>$b$</td>
</tr>
</tbody>
</table>

Rules: 1. No communication after inputs received
2. They \textit{win} if $a \oplus b = s \land t$

With classical resources, \( \Pr[a \oplus b = s \land t] \leq 0.75 \)

But, with prior entanglement state $|00\rangle - |11\rangle$,
\[
\Pr[a \oplus b = s \land t] = \cos^2(\pi/8) = \frac{1}{2} + \frac{1}{4}\sqrt{2} = 0.853\ldots
\]
The quantum strategy

- Alice and Bob start with entanglement
  \[ |\phi\rangle = |00\rangle - |11\rangle \]

- **Alice:** if \( s = 0 \) then rotate by \( \theta_A = -\pi/16 \)
  else rotate by \( \theta_A = +3\pi/16 \) and measure

- **Bob:** if \( t = 0 \) then rotate by \( \theta_B = -\pi/16 \)
  else rotate by \( \theta_B = +3\pi/16 \) and measure

\[
\cos(\theta_A - \theta_B ) (|00\rangle - |11\rangle) + \sin(\theta_A - \theta_B ) (|01\rangle + |10\rangle)
\]

Success probability:
\[
\Pr[a \oplus b = s \land t] = \cos^2(\pi/8) = \frac{1}{2} + \frac{1}{4}\sqrt{2} = 0.853\ldots
\]
**Nonlocality in operational terms**

- Information processing task
  - Classically, communication is *needed*
  - Quantum entanglement

![Diagram showing the relationship between information processing task and classical communication vs. quantum entanglement.](image)
The magic square game
**Magic square game**

**Problem:** fill in the matrix with bits such that each row has even parity and each column has odd parity

$$
\begin{array}{ccc}
  a_{11} & a_{12} & a_{13} & \text{even} \\
  a_{21} & a_{22} & a_{23} & \text{even} \\
  a_{31} & a_{32} & a_{33} & \text{even} \\
\end{array}
$$

odd odd odd

**Game:** ask Alice to fill in one row and Bob to fill in one column

They **win** iff parities are correct and bits agree at intersection

**Success probabilities:** $8/9$ classical and 1 quantum

[Mermin, 1990]
Distance measures for quantum states
Distance measures

Some simple (and often useful) measures:

• **Euclidean distance:** \( || \psi - \varphi ||_2 \)

• **Fidelity:** \( | \langle \varphi | \psi \rangle | \)

Small Euclidean distance implies “closeness” but large Euclidean distance need not (for example, \( \psi \) vs \(-\psi\))

Not so clear how to extend these for mixed states …

… though fidelity does generalize, to \( \text{Tr} \sqrt{\rho^{1/2} \sigma \rho^{1/2}} \)
Trace norm – preliminaries (1)

For a normal matrix $M$ and a function $f: \mathbb{C} \rightarrow \mathbb{C}$, we define the matrix $f(M)$ as follows:

$$M = U^\dagger D U,$$

where $D$ is diagonal (i.e. unitarily diagonalizable).

Now, define $f(M) = U^\dagger f(D) U$, where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix}, \quad f(D) = \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_d) \end{bmatrix}.$$
Trace norm – preliminaries (2)

For a normal matrix $M = U^* D U$, define $|M|$ in terms of replacing $D$ with

$$|D| = \begin{bmatrix}
|\lambda_1| & 0 & \ldots & 0 \\
0 & |\lambda_2| & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & |\lambda_d|
\end{bmatrix}$$

This is the same as defining $|M| = \sqrt{M^* M}$ and the latter definition extends to all matrices (not necessarily normal ones), since $M^* M$ is positive semidefinite
Trace norm/distance – definition

The trace norm of $M$ is $\|M\|_{\text{tr}} = \|M\|_1 = \text{Tr}|M| = \text{Tr}\sqrt{M^\dagger M}$

Intuitively, it’s the 1-norm of the eigenvalues (or, in the non-normal case, the singular values) of $M$

The trace distance between $\rho$ and $\sigma$ is defined as $\|\rho - \sigma\|_{\text{tr}}$

Why is this a meaningful distance measure between quantum states?

**Theorem:** for any two quantum states $\rho$ and $\sigma$, the optimal measurement procedure for distinguishing between them succeeds with probability $\frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_{\text{tr}}$
Distinguishing between two arbitrary quantum states
Holevo-Helstrom Theorem (1)

Theorem: for any two quantum states $\rho$ and $\sigma$, the optimal measurement procedure for distinguishing between them succeeds with probability $\frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_\text{tr}$ (equal prior probs.)

Proof* (the attainability part):

Since $\rho - \sigma$ is Hermitian, its eigenvalues are real

Let $\Pi_+$ be the projector onto the positive eigenspaces

Let $\Pi_-$ be the projector onto the non-positive eigenspaces

Take the POVM measurement specified by $\Pi_+$ and $\Pi_-$ with the associations $+ \equiv \rho$ and $- \equiv \sigma$

* The other direction of the theorem (optimality) is omitted here
Holevo-Helstrom Theorem (2)

Claim: this succeeds with probability $\frac{1}{2} + \frac{1}{4} \|\rho - \sigma\|_{tr}$

Proof of Claim:

A key observation is $\text{Tr}(\Pi_{+} - \Pi_{-}) (\rho - \sigma) = \|\rho - \sigma\|_{tr}$

The success probability is $p_s = \frac{1}{2} \text{Tr}(\Pi_{+} \rho) + \frac{1}{2} \text{Tr}(\Pi_{-} \sigma)$

& the failure probability is $p_f = \frac{1}{2} \text{Tr}(\Pi_{+} \sigma) + \frac{1}{2} \text{Tr}(\Pi_{-} \rho)$

Therefore, $p_s - p_f = \frac{1}{2} \text{Tr}(\Pi_{+} - \Pi_{-}) (\rho - \sigma) = \frac{1}{2} \|\rho - \sigma\|_{tr}$

From this, the result follows □
Purifications & Ulhmann’s Theorem

Any density matrix $\rho$, can be obtained by tracing out part of some larger pure state:

$$\rho = \sum_{j=1}^{d} \lambda_j \varphi_j \langle \varphi_j | = \text{Tr}_2 \left( \sum_{j=1}^{m} \sqrt{\lambda_j} \varphi_j \right) \left( \sum_{j=1}^{m} \sqrt{\lambda_j} \langle \varphi_j | \langle j | \right)$$

a purification of $\rho$

Ulhmann’s Theorem*:

The fidelity between $\rho$ and $\sigma$ is the maximum of $\langle \varphi | \psi \rangle$ taken over all purifications $|\psi\rangle$ and $|\varphi\rangle$

* See [Nielsen & Chuang, pp. 410-411] for a proof of this

Recall our previous definition of fidelity as

$$F(\rho, \sigma) = \text{Tr} \sqrt{\rho^{1/2} \sigma \rho^{1/2}} = \|\rho^{1/2} \sigma^{1/2}\|_{tr}$$
Relationships between fidelity and trace distance

\[ 1 - F(\rho, \sigma) \leq \|\rho - \sigma\|_{tr} \leq \sqrt{1 - F(\rho, \sigma)^2} \]

See [Nielsen & Chuang, pp. 415-416] for more details